Kurdish Studies
December 2022
Volume: 10, No: 2, pp 1260-1266
ISSN: 2051-4883 (Print) | ISSN 2051-4891 (Online)
www.KurdishStudies.net

DOI: 10.53555/ks.v10i2.4062

An Investigation And Examination Of Minimum Connected Domination In Fuzzy Graphs

Dr A J Khan[1*], Naziya Mirza[2]

[1*]Professor Mathematics, MATS University, Raipur, khanaj@matsuniversity.ac.in
[2]Research Scholar, MATS University, Raipur, naziyamirza07@gmail.com

Abstract
This study explores the concept of minimum connected domination in fuzzy graphs, focusing on defining and analyzing key parameters such as adjacent dominating sets and independent domination sets. Through the examination of fuzzy semi graphs, particularly in the context of end, strong, and consecutive domination, the research contributes to the theoretical understanding of domination theory within fuzzy frameworks. Despite its theoretical rigor, the study identifies limitations in practical applications and the absence of efficient algorithms for real-world implementation. Future research directions are proposed to investigate dynamic fuzzy semi graphs, develop computational methods, and apply these theoretical concepts to practical scenarios, thereby enhancing their relevance and utility in various fields.

Keywords: *Fuzzy Graphs, Minimum Connected Domination, Dominating Sets, Fuzzy Semi graphs.*

1. Introduction

Domination in graph theory is a fundamental concept that has significant implications in various fields, including computer science, network theory, and operations research. It refers to a set of vertices within a graph such that every vertex in the graph is either included in this dominating set or is adjacent to at least one member of the set. This concept is crucial for ensuring connectivity and efficiency in network structures. With the advent of fuzzy logic, the exploration of domination in fuzzy graphs has emerged as a vital area of research. Fuzzy graphs extend the classical graph theory by incorporating degrees of uncertainty and partial truth, making them particularly relevant in real-world applications where relationships are not always binary. This study focuses on minimum connected domination in fuzzy graphs, specifically by defining and analyzing key parameters such as adjacent dominating sets and independent domination sets. By examining fuzzy semi graphs and considering different types of domination—including end, strong, and consecutive domination—this research aims to contribute to a deeper theoretical understanding of domination theory within fuzzy frameworks. While the theoretical developments presented in this study are significant, they also highlight limitations in practical applications, particularly concerning the absence of efficient algorithms for real-world implementation. These challenges underscore the need for further investigation into dynamic fuzzy semi graphs and the development of computational methods. Such efforts will not only advance theoretical knowledge but also enhance the applicability of these concepts across diverse fields, thereby ensuring that domination theory remains relevant in an increasingly complex and uncertain world.

2. Literature Review

The study of minimum connected domination in fuzzy graphs is a significant area of research that combines concepts from fuzzy set theory and graph theory. This investigation focuses on understanding how fuzzy graphs, with their inherent uncertainties, can be efficiently dominated while ensuring connectivity. Various approaches have been proposed, addressing applications in optimization, decision-making, and network design, making it a critical field in theoretical and applied mathematics.

Literature Summary

Authors	Work Done	Findings
Wang & Zhu (2022)	Developed a graph reduction method in a path information-based rough directed graph model	Proposed an efficient method to reduce complex graph structures while preserving essential path information
Enriquez et al. (2021)	Explored domination concepts in fuzzy directed graphs	Introduced novel techniques for domination in fuzzy directed graphs, enhancing understanding of control in fuzzy networks
Atef et al. (2021)	Investigated various fuzzy graph models	Presented innovative algorithms for fuzzy graph applications in decision-making and network analysis
Koam et al. (2020)	Conducted decision-making analysis using fuzzy graph structures	Provided new insights into decision-making processes using fuzzy graph structures for practical applications

Akram & Zafar (2019a)	Developed measures of connectivity in rough fuzzy network models	Introduced advanced connectivity measures in rough fuzzy networks, improving model accuracy and applicability
Akram & Zafar	Explored rough fuzzy digraphs with real-world applications	Showed how rough fuzzy digraphs could be applied to practical scenarios, including
(2019b)		decision-making and network analysis
Sitara et al. (2019)	Analyzed fuzzy graph structures with various applications	Demonstrated the utility of fuzzy graph structures in solving real-world problems, particularly in optimization and control
Akram & Arshad (2018a)	Developed fuzzy rough graph theory with a focus on applications	Established a foundation for fuzzy rough graph theory, offering insights into its applications in network modeling
Akram & Arshad (2018b)	Introduced a new approach using fuzzy rough digraphs for decision making	Applied fuzzy rough digraphs in decision making scenarios, enhancing the efficiency of solutions in uncertain environments
Mordeson et al. (2018)	Conducted foundational research in fuzzy graph theory and its extensions	Provided a comprehensive understanding of fuzzy graph theory, laying the groundwork for future research and applications

3. Research Gap

Despite advancements in fuzzy graph domination, critical research gaps persist. There is a lack of empirical studies validating theoretical constructs, particularly regarding adjacent dominating sets and independent domination sets, without efficient algorithms for real-world applications. The literature also inadequately explores dynamic fuzzy semi graphs, limiting understanding of evolving domination in changing conditions. Furthermore, there is a need for computational methods to handle complexities in large-scale networks. Integrating fuzzy domination concepts with emerging technologies like AI presents opportunities to enhance relevance and applicability across diverse fields.

4. Problem Statement

This study addresses the challenges of minimum connected domination in fuzzy graphs, focusing on the lack of efficient algorithms for practical implementation. It aims to define key parameters and explore dynamic fuzzy semi graphs to enhance both theoretical understanding and real-world applicability.

5. Methodology

This study introduces the concept of strong arc fuzzy semi graphs and analyzes parameters related to dominating and independent sets. It defines minimum and maximum adjacent dominating numbers, as well as end node and consecutive dominating numbers, while also examining end node, strongly, and consecutive independent dominating numbers. These parameters reflect the minimum and maximum cardinalities of dominating sets.

Domination in Fuzzy Semi graphs:

1. **Definition:** Let G = (σ, μ, η) be a fuzzy semi graph with nodes u and v. If there exists a strong arc from u to **v,** we define that u dominates v.
2. **Definition:** An adjacent dominating set (ad-set) in the fuzzy semi graph G, denoted as $D_a \subseteq V$, is defined as a set where for every $v \in V - D_a$ there exists a node $u \in D_a$ such that u is adjacent to v. The minimum cardinality of an ad-set is denoted as $\gamma_a(G)$, while the maximum cardinality is denoted as $\Gamma_a(G)$.
3. **Definition:** An end node adjacent dominating set (ad -set) in the fuzzy semi graph G, denoted as $D_e \subseteq V$, satisfies the following conditions: (i) D is an ad-set, and, (ii) each end node $v \in V - D_e$ ∈adjacent to some end node $u \in D_e$ in G. The minimum cardinality of an adset is denoted as $\gamma_a(G)$, while the maximum cardinality is denoted as $\Gamma_a(G)$.
4. **Definition:** A consecutive adjacent dominating set (cad-set) in a fuzzy semi graph G, denoted as $D_{ca} \subseteq V$, is defined as a set in which, for every $v \in V - D_{ca}$, there exists a $u \in D_{ca}$ such that u is consecutively adjacent to v in G. The minimum cardinality of a cad-set is denoted as $\gamma ca(G)$, and the maximum cardinality is denoted as $\Gamma ca(G)$.
5. **Definition:** Let G=(σ, μ, η) be a fuzzy semi graph. The adjacent degree of a node v is defined as follows:

$$d_a(u) = \sum \sum_{e \in E} \mu(e).$$

The minimum degree of G is denoted as $\delta_a = \wedge\{d_a/v \in V\}$, and the maximum degree of G is denoted as Δ_a - $\vee\{da/v \in V\}$.

6. **Definition:** Let $G=(\sigma, \mu, \eta)$ be a fuzzy semi graph. The end adjacent degree of a node v is defined as:

$$d_{ea}(u) = \sum \sum_{e \in E} \mu(e).$$

The minimum degree of G is denoted as δ_{ea} - $\wedge\{d_{ea}/v \in V\}$, and the maximum degree of G is denoted as Δ_{ea} - $\vee\{d_{ea}/v \in V\}$.

7. **Definition:** Let $G=(\sigma, \mu, \eta)$ be a fuzzy semi graph. The consecutive adjacent degree of a node v is defined as:

$$d_{ca}(u) = \sum \mu(uv).$$

The minimum degree of G is denoted as δ_{ca} - $\wedge\{d_{ca}/v \in V\}$, and the maximum degree of G is denoted as Δ_{ca} - $\vee\{d_{ca}/v \in V\}$.

Independent Domination in Fuzzy Semi graphs:

1. **Definition:** Let $G = (\sigma, \mu, \eta)$ be a fuzzy semi graph. A set $S \subseteq V$ is considered fuzzy independent if there are no strong arcs among its elements. The set S_e is termed end-independent if no two end nodes belong to an edge in $S_e \subseteq V$. The set S_s is defined as strongly independent if no two adjacent nodes are present in $S_s \subseteq V$. The set S_{ca} is considered ca-independent if no two nodes are consecutively adjacent in $S_{ca} \subseteq$ V.
2. **Definition:** The minimum cardinality of an end-independent dominating set is referred to as the e-independence domination number, denoted as $I_e(G)$. Similarly, we can define i_s (G) and I_{ca} (G).
3. **Definition:** The maximum cardinality of an end-independent dominating set is also called the e-independence domination number, denoted as $\beta_e(G)$. Similarly, we can define $\beta_s\{G\}$ β_{ca} (G).

6. Limitation

- Limited exploration of domination types.
- No practical applications or real-world validation.
- Absence of efficient algorithms for domination numbers.
- Simplistic degree definitions.
- Focus solely on strong arcs.
- Excludes dynamic or temporal fuzzy semi graphs.

7. Results and Discussion

Example 1: Let G = (σ, μ, η) be a fuzzy semi graph, as illustrated in Figure

- Minimal adjacent dominating set Da = {P, S}.
- Minimum adjacent domination number γ_a = 0.5+0.5=1.
- Maximal adjacent dominating set D_a = {Q, U}
- Maximum adjacent domination number Γ_a = 0.7+0.6=1.3.
- Minimum adjacent degree δ={P}=0.5.
- Maximum adjacent degree Δ={S}=0.9.

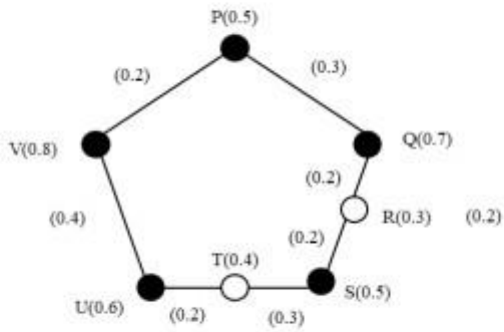

Fig 1 Adjacent dominating set.

Example 2: Let $\mathcal{G} : (\sigma, \mu,\ )$ be a fuzzy semi graph as shown in Figure 2.

- Minimal End adjacent dominating set D_{ea} = {P, T}.
- Minimum End adjacent domination number γea = {0.5+0.8} = 1.3.
- Maximal End adjacent dominating set D_{ea} = {P, R, T, V}.
- Maximum End adjacent domination number Γea = {0.5+0.6+0.8+0.9} = 2.8.
- Minimum End adjacent degree $\delta = \{Q\}$ = 0.4.

Maximum End adjacent degree $\Delta = \{T\}$ = 0.8.

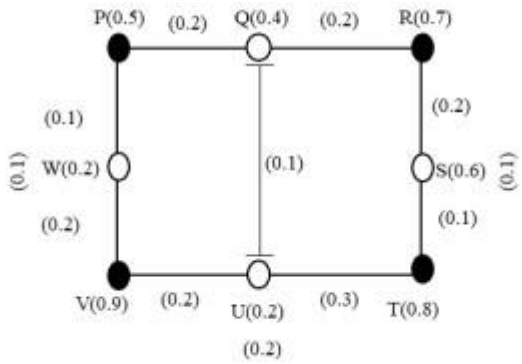

Fig 2 End adjacent dominating set.

Example 3: Let $\mathcal{G} : (\sigma,,\eta)$ be a fuzzy semi graph as shown in Figure 3.

- Minimal Consecutive adjacent dominating set D_{ca} = {Q,S,U}.
- Minimum Consecutive adjacent domination number γca = {0.2+0.3+0.4} = 0.9.
- Maximal Consecutive adjacent dominating set D_{ca} = {P,S,V}.
- Maximum Consecutive adjacent domination number Γca = {0.4+0.3+0.8} = 1.5.
- Minimum Consecutive adjacent degree δ = { } = 0.2.

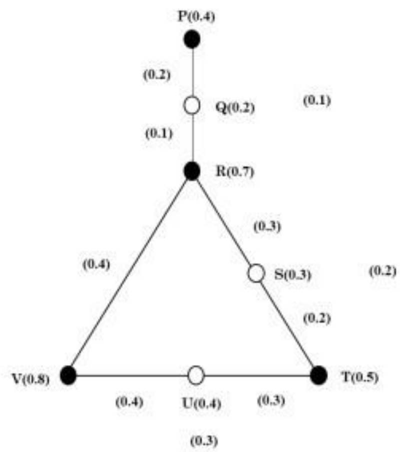

Fig 3 Consecutive adjacent dominating set.

Maximum Consecutive adjacent degree $\Delta = \{V\} = 0.8$.
Definition: Let $\mathcal{G} : (\sigma,,\eta)$ be a fuzzy semi graph. A node $u \in D$ is said to be a-isolates in D, if no v in D such that u and v are adjacent.

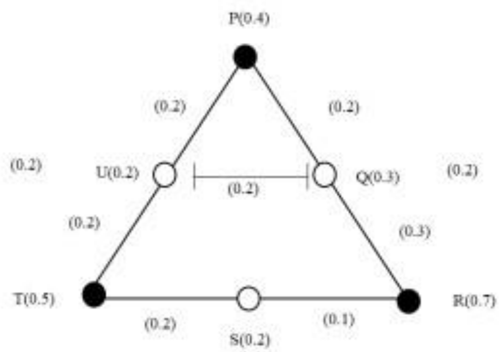

Figure 4 Probability Distribution Diagram for a Three-Variable System.

Here $D = \{P, Q, S\}$, S is an a-isolate, but P & Q are not a-isolates.

Observation: For any given fuzzy semi graph $\mathcal{G}$,

- $\gamma a\ (\mathcal{G}) \le \gamma ea\ (\mathcal{G})$
- $\gamma a\ (\mathcal{G}) \le \gamma ca\ (\mathcal{G})$
- $\gamma ea(\mathcal{G}) \le \gamma ca(\mathcal{G})$

Observation: Let P_k represent a path fuzzy semi graph and C_k denote a cycle fuzzy semi graph with *k* nodes, consisting of *n* end nodes and *m* middle nodes, where $m + n=k$. Define $G_k = P_k$ or C_k. Then, we have the following inequalities:

- $\gamma_a (G) \leq \gamma_{ea} (G) \leq [\frac{k}{2}]$
- $\gamma_{ca} (G) \leq [\frac{k}{2}]$

Theorem 1: An adjacent dominating set (ad-set) D_a of a fuzzy semi graph *G* is considered minimal if and only if, for each node $u \in D_a$, at least one of the following properties holds:
• *u* is an adjacent-isolate node of D_a.
• There exists a node $v \in V-D_a$ such that $N_a(v) \cap D_a = \{u\}$.

Proof:
Assume D_a is a minimal ad-set of *G*. For every node *u* in D_a, consider the set $D_a' = D_a - \{u\}$ which is not an ad-set. Consequently, there must be a node $v \in V - D_a'$ that is not adjacent to at least one node in D_a'.
Now, we examine two scenarios: either $v, u \in D_a$ *or* $v\ u \in V - D_a$.
• If $v, \in D_a$, then *u* is an adjacent-isolate node of D_a.
• If $v \in V - D_a$ and is not adjacent dominated by D_a' but is adjacent dominated by D_a,
then $u \in D_a$ must be the only strong neighbor of *v* in $V-D_a$, meaning $N_a(v) \cap D_a$ - $\{u\}$.
Conversely, assume that D_a is an ad-set and that for each node u ∈ Da, one of the stated conditions holds. We need to demonstrate that D_a is a minimal ad-set.
Suppose D_a is not a minimal ad-set, meaning there exists a node $u \in D_a$ such that D_a' is still an ad-set. Consequently, *u* must be a strong neighbor to at least one node in D_a', which contradicts condition.
Additionally, if D_a' is an ad-set, then every node in $V-D_a$ must be a strong neighbor to at least one node in D_a', contradicting condition.
Thus, we reach a contradiction, indicating that neither condition nor condition can fail.
Example 4: In Figure 4, the set $\{P, Q, S\}$ is identified as a minimal ad-set where its nodes satisfy conditions and. Within this set, node *S* meets condition, while nodes *P* and *Q* satisfy condition as stated in Theorem.
Theorem 2: For any fuzzy semi graph *G,* it holds that $\gamma_a(G) \leq k - \Delta_a(G)$.
Proof: Let *v* be a node of maximum adjacent degree $\Delta_a(G)$. Then *v* is adjacent to $N_a(v)$ nodes, implying $\Delta_a(G) \leq |N_a(v)|$. Therefore, $V-N_a(v)$ forms an ad-set. This leads us to the conclusion that $\gamma_a(G) \leq |V-N_a(v)|$, which results in $\gamma_a(G) \leq k-\Delta_a(G)$.
Theorem 3: It *G* is a fuzzy semi graph and D_a is a minimal ad-set, then the complement of D_a, denoted $V- D_a$, is an ad-set.
Proof: Assume $V- D_a$ is not an adjacent dominating set. This implies there exists a node $u \in - D_a$ that is not adjacent to any node in $V- D_a$. Since D_a is a minimal ad-set, every node in $V- D_a$ must be adjacent to at least one node in D_a . This creates a contradiction, as it suggests that *u* is not adjacent to any node in $V- D_a$, while each node in $V- D_a$ must be adjacent to some node in D_a. Therefore, our initial assumption that $V- D_a$ is not an ad-set is false.
Consequently, $V- D_a$ is indeed an ad-set.

Example 5: Let $\mathcal{G} : (\sigma,,\eta)$ be a fuzzy semi graph as shown in Figure 5

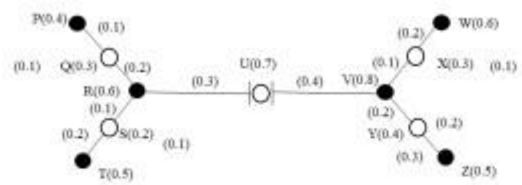

Fig 5 e-independent set.

• Maximal e-independent set $Se = \{P, T, U, W, Z\}$.
• Maximum e-independence number $\beta e = \{0.4+0.5+0.7 +0.6+0.5\} = \{2.7\}$.
• Minimal e-independent set $Se = \{R, V\}$.
• Minimum e-independence number $ie = \{0.6+0.8\} = \{1.4\}$.
• Minimum e-independent degree $\delta = \{R\} = \{0.2\}$. • Maximum e-independent degree $\Delta = \{U\} = \{0.7\}$.

Example 6: Let $\mathcal{G} : (\sigma, \mu, \eta)$ be a fuzzy semi graph as shown in Figure 6.

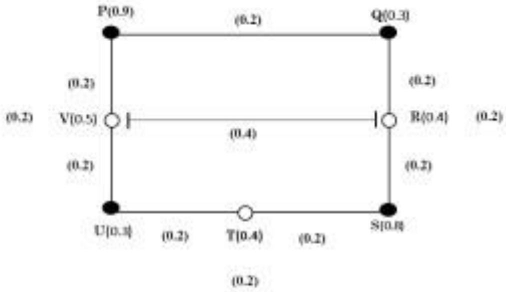

Fig. 6 Strongly independent set Maximal.

- Strongly independent set $SS = \{P, R, T\}$.
- Maximum strongly independence number βS = {0.9+0.4+0.4} = 1.7.
- Minimal strongly independent set $SS = \{Q, U\}$.
- Minimum strongly independence number is = {0.3+0.3} = 0.6.
- Minimum strongly independent degree $\delta = \{P\}$ = 0.4. • Maximum strongly independent degree $\Delta = \{R\}$ = 0.6.

Example 7: Let $\mathcal{G} : (\sigma, \mu, \eta)$ be a fuzzy semi graph as shown in Figure 7.

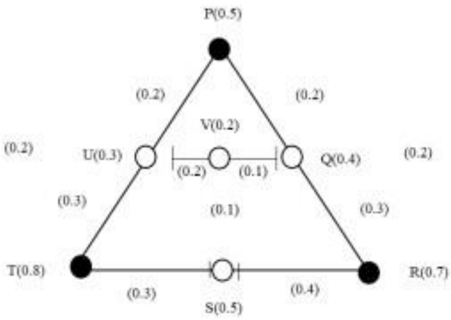

Fig 7 Consecutive adjacent independent set.

- Maximal Consecutive adjacent independent set $Sca = \{P, V, T, R\}$.
- Maximum Consecutive adjacent independence number βca = {0.5+0.2+0.8+0.7} = {2.2}.
- Minimal Consecutive adjacent independent set $Sca = \{P, V, S\}$.
- Minimum Consecutive adjacent independence number ica = {0.5+0.2+0.5} = {1.2}.
- Minimum Consecutive adjacent independent degree $\delta = \{V\}$ = {0.2}.
- Maximum Consecutive adjacent independent degree $\Delta = \{S\}$ = {0.7}

Observation: For any given fuzzy semi graph $\mathcal{G}$,
- $\beta e\,(\mathcal{G}) \leq \beta s\,(\mathcal{G})$
- $\beta s\,(\mathcal{G}) \leq \beta ca(\mathcal{G})$
- $\beta e\,(\mathcal{G}) \leq \beta ca(\mathcal{G})$

Observation: Let P_k represent a path fuzzy semi graph and C_k a cycle fuzzy semi graph with k nodes, where n are end nodes and mmm are middle nodes, such that $m + n = k$. Let $G_k = P_k$ *or* C_k. *Then:*
- $\beta_e\,(G) \leq \beta_s\,(G) \leq [^k]$.

2 • $\beta_{ca}\,(G) \leq [^k]$.
2

Theorem 4: A path fuzzy semi graph G is a maximal fuzzy *e*-independent set if and only if it satisfies the conditions of both a fuzzy e-independent set and a fuzzy *ea*-dominating set.

Proof: Consider a path fuzzy semi graph G, and let S_e be a maximal fuzzy e-independent set. For every node $u \in V - S_e$, the set $S_e \cup \{u\}$ is not fuzzy e-independent. This means that for every node $u \in V - S_e$, there exists a node $v \in S_e$ such that (u, v) is a strong neighbor. Therefore, S_e is an ea-dominating set, meaning S_e is both fuzzy e-independent and eadominating. Conversely, suppose S_e is both fuzzy *e*-independent and e_a-dominating. We need to show that it is a maximal fuzzy *e*-independent set. Assume S_e is not maximal. Then, there exists a node $u \in V - S_{eu}$ such that $Se \cup \{u\}$ is fuzzy *e*-independent. However, if $S_e \cup \{u\}$ is fuzzy *e*-independent, then no node in S_e is a strong neighbor of u, which contradicts the fact that S_e is e_a-dominating. Hence, S_e is a maximal fuzzy *e*-independent set.

Theorem 5: In a fuzzy semi graph G, every maximal strongly independent set is a minimal added-set in G.

Proof: The proof follows similarly to the proof of previous Theorem.

Observation: For any fuzzy semi graph *G,*

$\gamma a \leq \gamma ea \leq \gamma ca \leq \beta e \leq \beta s \leq \beta ca$.
Proof: Refer to examples [1, 2, 3…].

8. Conclusion

This study provides a detailed investigation into the concept of minimum connected domination in fuzzy graphs, defining key parameters such as adjacent dominating sets, independent domination sets, and their variations. By analyzing fuzzy semi graphs through end, strong, and consecutive domination, it enhances understanding of domination theory in a fuzzy context. However, the research primarily focuses on theoretical constructs, with limited real-world applications and no efficient algorithms presented for practical implementation. Future work should explore dynamic fuzzy semi graphs, develop computational methods, and apply these concepts to real-world scenarios for broader applicability.

Future Scope

- **Dynamic Fuzzy Semi graphs:** Study fuzzy semi graphs in dynamic environments to model real-world scenarios.
- **Algorithm Development:** Create efficient algorithms for computing minimum connected domination sets.
- **Real-World Applications:** Apply fuzzy domination concepts in fields like network security and bioinformatics.
- **Optimization Techniques:** Use optimization methods, such as machine learning, for solving fuzzy domination problems.
- **Extended Variations:** Explore other fuzzy graph types, like weighted or probabilistic graphs.

Suggestions

- **Cross-Disciplinary Collaboration:** Engage with researchers in fields like computer science, biology, and engineering to explore interdisciplinary applications of fuzzy graph theory.
- **Computational Tools:** Develop software tools to model and analyze fuzzy graphs, improving the usability and application of theoretical findings.
- **Real-World Data Testing:** Test the fuzzy domination concepts on real-world datasets to validate their practicality and effectiveness in solving tangible problems.
- **Algorithmic Development:** Develop efficient algorithms for computing minimum connected domination sets in fuzzy graphs, making the concepts more practical and accessible.

References

1. Wang, D., & Zhu, P. (2022). Graph reduction in a path information-based rough directed graph model.
2. Enriquez, et al. (2021). Domination in fuzzy directed graphs.
3. Atef, M., Atik, E., El Fattah, A., & Nawar, A. (2021).
4. Koam, A., Akram, M., & Liu, P. (2020). Decision-making analysis based on fuzzy graph structure.
5. Akram, M., & Zafar, F. (2019a). A new approach to compute measures of connectivity in rough fuzzy network models.
6. Akram, M., & Zafar, F. (2019b). Rough fuzzy digraphs with application. Journal of Applied Mathematics and Computation, 59, 91–127.
7. Sitara, M., Akram, M., & Bhatti, M. (2019). Fuzzy graph structures with application. Mathematics, 7.
8. Akram, M., & Arshad, M. (2018a). Fuzzy rough graph theory with applications.
9. Akram, M., & Arshad, M. (2018b). A new approach based on fuzzy rough digraphs for decision-making.
10. Mordeson, J., Mathew, S., & Malik, D. (2018).