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Exact Solutions and Classification of the (1+3) Dimensional Generalized Modified Schrödinger Equation Using Symmetry Reduction Approach

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Abstract
This study focuses on nonlinear equations, particularly the (1 + 3) dimensional generalized modified Schrödinger equation (GMSE) as a key example. Given the extensive use of classical Lie symmetry methods, the research applies Lie symmetry analysis to explore the (GMSE) in detail. Lie symmetries of the equation are derived to identify rare classes of exact solutions, with the arbitrary functions in each solution offering a wide range of possible solution profiles. The Lie symmetry method holds considerable future potential for generating more diverse solutions, as it allows solutions to incorporate functions and arbitrary constants. This work also effectively highlights the uniqueness of the solutions when compared to previously published results.

Keywords: Exact solutions, Classification, Schrödinger equation, Symmetry reduction.

1 Introduction

A broad range of physical phenomena and behaviors can be modeled by nonlinear dispersive partial differential equations (PDEs). Differentiating the dispersive effects of these equations is essential. Equations can display linear or non-linear dispersive effects, or their dispersive effects may completely vanish. The investigation into symmetries of PDEs is advancing, as they are essential for expanding known solutions, realizing linearization, and finding new integrable PDEs. Physical existence or models are often described using nonlinear equations. Analytical solutions of nonlinear PDEs are crucial in areas like mathematical physics, fluid dynamics, nonlinear optics, plasma physics, modeling and engineering. In recent decades, considerable attention has been given to extracting solutions of nonlinear systems, such as traveling wave solutions, solitary waves, periodic waves, kink and anti-kink solutions, solitons, and studying the integrability of certain fascinating nonlinear partial differential equations (PDEs) [1–5]. This includes equations like the Burgers equation [6], modified equal width-Burgers equation [7], combined KdV-mKdV equation [8], nonlinear Schrödinger equation [9], the KdV model [10], and the modified equal width equation [11]. Some results were derived using the rational sinhcosh methods and polynomial function. Hence, determining exact solutions has gained essential importance in the field of nonlinear science. In recent years, physicists and mathematicians have extensively worked on this subject, introducing numerous useful techniques, including the mapping method [12], the reciprocal Backlund transformation method [13], Hirota's bilinear method [14], the Painleve expansion [15], the homogeneous balance approach [16], the Expfunction technique [17], the rational expansion method [18], the Jacobi elliptic function expansion method [19,20], and many other approaches [21,22]. The application of the Lie symmetry method to PDEs provides a robust approach for deriving reductions and invariant solutions [23]. To describe symmetries, Lie symmetry groups serve as a major family of mathematical tools [24].

Sophus Lie is widely recognized by mathematicians as the founder of the theory of transformation groups, which led to the development of the modern theory of Lie groups [25]. Recently, Tian et al. [26] successfully developed a direct and effective approach for the symmetry-preserving discretization of generalized higher-order equations and proposed an open problem regarding the symmetries and multipliers of conservation laws. We analyze (3.1) in this paper from the viewpoint of symmetry reductions in PDEs, obtaining the Lie point symmetries admitted by (3.1) for arbitrary functions to derive exact solutions. The mathematical technique of symmetry analysis [27,28] is used to examine and understand the symmetries that exist within equations or physical systems. Symmetries describe the invariance of systems under specific transformations. In symmetry analysis of differential equations, the goal is to identify transformations, typically generated by Lie groups, that maintain the equation's form or transform it into an equivalent one. The symmetries can expose important details about the equations, potentially leading to more efficient solutions. Lie theory is of great significance in nonlinear and engineering fields, with countless practical uses. In contemporary research, a significant volume of literature on Lie theory can be found [29–31]. Kumar and Kumar [32] employ the Lie symmetry method to find the analytical solutions of the (2 + 1)-dimensional modified CBS equation. The solutions to many well-known nonlinear PDEs are achievable solely through symmetry methods. Lie point symmetries are instrumental in identifying similarity transformations, a method that researchers use to introduce new dependent and independent variables. By introducing similarity variables into PDEs, the number of independent variables is reduced, although the order of the PDEs remains constant. A sequence of reductions applied to the simplified PDEs ultimately leads to ODEs. By repeatedly applying the Lie symmetry technique on the reduced ODEs, lower-order ODEs are derived, from which particular solutions to the given system can be computed. Currently, researchers rely on powerful Computer Algebra Systems (CAS), like Maple and Mathematica (commercial), to efficiently carry out symmetry calculations. To calculate the symmetries in this work, we make use of the Mathematica and Maple [33,34]. We employ symmetry approaches to achieve our goals, as we are focused on obtaining exact solutions to the nonlinear GMSE (3.1).

2 Fundamental Operators

To utilize the Lie method for GMSE (3.1) we examine a one-parameter Lie group of point transformations acting on the dependent and independent variable spaces.

$$\begin{aligned} u' &= u + \epsilon\lambda(u,t,x,y,z) + O(\epsilon^2), \\ t' &= t + \epsilon\psi(u,t,x,y,z) + O(\epsilon^2), \\ x' &= x + \epsilon\xi(u,t,x,y,z) + O(\epsilon^2), \\ y' &= y + \epsilon\phi(u,t,x,y,z) + O(\epsilon^2), \\ z' &= z + \epsilon\eta(u,t,x,y,z) + O(\epsilon^2), \end{aligned} \quad (2.1)$$

related to the infinitesimal generators

$$W = \lambda(u,t,x,y,z)\partial_u + \psi(u,t,x,y,z)\partial_t + \xi(u,t,x,y,z)\partial_x + \phi(u,t,x,y,z)\partial_y + \eta(u,t,x,y,z)\partial z. \quad (2.2)$$

In accordance with the symmetry invariant conditions [35–37], we stipulate that the second prolongation of the infinitesimal generator (2.2) leaves the solution space of PDEs (3.1) invariant, i.e.,

$$Pr^{[2]}Y(\frac{-\hbar^2}{2m}[u_{xx} + u_{yy} + u_{zz}] + \phi u = i\hbar u_t + f_u u_{tt} + f_{uu}u_t^2)|_{\Delta=0} = 0. \quad (2.3)$$

Where $\Delta = (\frac{-\hbar^2}{2m}[u_{xx} + u_{yy} + u_{zz}] + \phi u = i\hbar u_t + f_u u_{tt} + f_{uu}u_t^2)$, and considering $Pr^{[2]}Y$ as the second-order prolongation of the vector field (3.1), we split Eq. (2.3) with respect to the derivatives of u leading to an overdetermined system of equations for the infinitesimals $\lambda(u,t,x,y,z)+\psi(u,t,x,y,z)+ \xi(u,t,x,y,z) + \phi(u,t,x,y,z) + \eta(u,t,x,y,z)$, and the functions f_u and f_{uu}. The following equation is obtained from (3.6)-(3.11). In the next section, the second-order prolongation is applied to find the Lie point symmetries and exact solutions of Eq. (3.1).

3 Classification and Exact Solutions for the (1 + 3) Dimensional Generalized Modified Schro¨dinger Equation (GMSE).

$$\frac{-\hbar^2}{2m}[u_{xx} + u_{yy} + u_{zz}] + \phi u = i\hbar u_t + f_u u_{tt} + f_{uu}u_t^2. \quad (3.1)$$

In this section, we prompt the whole classification of the classical Lie symmetry and exact solution of (1 + 3) dimensional GMSE. The given equation is

$$\frac{-\hbar^2}{2m}[u_{xx} + u_{yy} + u_{zz}] + \phi u - i\hbar u_t - f_u u_{tt} - f_{uu}u_t^2 = 0, \quad (3.2)$$

where the function f_u and f_{uu} are arbitrary.

- ϕ is a potential.
- u is a wave function.
- $\hbar$ is a plank's constant, $\hbar = \frac{h}{2\pi}$.
- m is the particle mass.
- ι a mathematical quantity which is called "imaginary number", which is equal to $\sqrt{-1}$.
- t is time, which describe how "u" modify its structure over time..

In a vector fields LPS are

(3.3)

$$Y = \phi(u,t,x,y,z)\frac{\partial}{\partial u} + \xi^1(u,t,x,y,z)\frac{\partial}{\partial t} + \xi^2(u,t,x,y,z)\frac{\partial}{\partial x} + \xi^3(u,t,x,y,z)\frac{\partial}{\partial y} + \xi^4(u,t,x,y,z)\frac{\partial}{\partial z},$$

$\xi = \xi(u,t,x,y,z)$, $i = 1,2,3,4$, for above equation. The invariant condition is

$Q = \xi^1 u_t + \xi^2 u_x + \xi^3 u_y + \xi^4 u_z + \phi = 0.$ (3.4)

The Lie point symmetry generators of equation (3.2) are obtained By imposing the condition of invariance

$$Y^{[2]}[\frac{-\hbar^2}{2m}[u_{xx}+u_{yy}+u_{zz}]+\phi u - i\hbar u_t - f_u u_{tt} - f_{uu}u_t^2 = 0.] \,|_{(3.2)} = 0. \quad (3.5)$$

Where $Y^{[2]}$ is the invariant transformation's of 2nd order prolongation of Y and $|_{(3.2)}$ is the application of the surface conditions which are invariant. In above $Y^{[2]}$ can be derived, After expansion and separation with respect to the powers of various derivatives of u, a structure of linear PDEs with new coefficients is provided. ξ^1, ξ^2, ξ^3, ξ^4 and ϕ. From Eq (3.2), determining forms are

$\xi_u^1 = 0, \xi_u^3 = 0, \xi_u^2 = 0, \xi_{uu}^2 = 0, \xi_{uu}^1 = 0, \xi_{uu}^3 = 0, \xi_{ut}^1 = 0, \xi_{uu}^4 = 0, \xi_{ut}^2 = 0, \xi_{xu}^2 = 0, \xi_{tu}^3 = 0, \xi_{uy}^1 = 0, \xi_{xx}^3 = 0, \xi_{xx}^1 = 0, \xi_{xx}^4 = 0$, . (3.6)

The remaining determining equations are determined by using the values of the equation (3.6).

$$2\hbar\iota m\xi_t^2 - 2mf_u\xi_{xx}^2 - 4m\phi_x f_{uu} + 4mf_u\phi_{xu} - 2\hbar^2\phi_{xu} + h^2\xi_{xx}^2 + \hbar^2\xi_{tt}^2 = 0, \quad (3.7)$$

$$2\hbar\iota m\xi_t^3 - 2\hbar^2\phi_{yu} + h^2\xi_{tt}^3 + \hbar^2\xi_{xx}^2 + \hbar^2\xi_{yy}^3 + h^2\xi_{zz}^3 = 0, \quad (3.8)$$

$$-4m\phi_u f_{uu} + 4m\xi_x^2 f_{uu} - 2m\phi f_{uuu} = 0, \quad (3.9)$$

$$-\hbar^2\phi_u + 2mf_u\phi_u + 2h^2\xi_x^2 - 4mf_u\xi_x^2 + 2m\phi f_{uu} = 0, \quad (3.10)$$

$$-\hbar^2\phi_u + 2mf_u\phi_u + 2h^2\xi_x^2 - 4mf_u\xi_x^2 + 2m\phi f_{uu} = 0. \quad (3.11)$$

From equation (3.9)

$$\xi_x^2 = \frac{2m\phi f_{uuu} + 4m\phi_u f_{uu}}{4mf_{uu}}. \quad (3.12)$$

After taking the derivative of equation (3.11) and using values of equation (3.12), then Simplifying equation is

$$-4m\phi_u f_{uuu} + 4m\xi_{xu}^2 f_{uu} + 4m\xi_x^2 f_{uuu} - 2m\phi_u f_{uuu} - 2m\phi f_{uuuu} = 0. \quad (3.13)$$

Taking the derivative of equation (3.12) w.r.t.'u'.

$$\xi_{xu}^2 = \frac{\phi_u f_{uu} f_{uuu} + \phi f_{uu} f_{uuuu} - \phi f_{uuu} f_{uuu}}{2f_{uu}f_{uu}}. \quad (3.14)$$

Simplifying equation using the value of ξ_x^2 and ξ_{xu}^2 in equation (3.13).

$$2mf_{uuu}^2 f_{uuuu} - 8mf_{uu}f_{uuu}^2 f_{uuuu} + 4f_{uu}f_{uuu}f_{uuuu} + 8mf_{uu}^2 f_{uuu}^2 f_{uuuuu} - 16mf_{uu}f_{uuuu} - f_{uuuu} + 4mf_{uu}f_{uuuu} - f_{uuuuu} = 0. \quad (3.15)$$

Solving the remaining equation (3.15), involving the arbitrary function of the following form

$$f(u) = a\frac{u^2}{2}\ln(1+\tan u) + b\frac{u^2}{2}, \quad (3.16)$$

$$f(u) = \frac{ae^{bu}}{b^2} + \frac{cu^3}{6} + \frac{du^2}{2}. \quad (3.17)$$

The method for calculating Lie symmetry transformations is as follows, using equation (3.16) and taking the first and second derivatie's.

$$f_u = au\ln(1+\tan u) + \frac{a}{2}\ln(tan^2 u) + bu, \quad (3.18)$$

$$f_{uu} = a\ln(1+\tan u) + 2a\ln(\tan u) + b. \quad (3.19)$$

3.1 GMSE Lie Symmetries Involving $f(u) = a\frac{u^2}{2}\ln(1+\tan u) + b\frac{u^2}{2}$

If $f_u = au\ln(1+\tan u) + \frac{a}{2}\ln(tan^2 u) + bu$ and $f_{uu} = a\ln(1+\tan u) + 2a\ln(\tan u) + b$ then equation (3.2) becomes

$$\frac{-\hbar^2}{2m}[u_{xx}+u_{yy}+u_{zz}] + \phi u - i\hbar u_t - (au\ln(1+\tan u) + \frac{a}{2}\ln(tan^2 u) + bu)u_{tt} - (a\ln(1+\tan u) + 2a\ln(tanu) + b)u_t^2 = 0. \quad (3.20)$$

Simplifying the Eq (3.20), which gives five LPS

$$Y_1 = \frac{\partial}{\partial t}, Y_2 = \frac{\partial}{\partial x}, Y_3 = \frac{\partial}{\partial y}, Y_4 = \frac{\partial}{\partial z}, Y_5 = \frac{\partial}{\partial u}. \quad (3.21)$$

3.2GMSE Exact Solutions Involving $f(u) = a\frac{u^2}{2}\ln(1+\tan u) + b\frac{u^2}{2}\ln(tan^2 u) + bu$

Next, we obtain the ES of equation (3.20) using the aforementioned five LPS.

$$Y_1 = \frac{\partial}{\partial t}, Y_2 = \frac{\partial}{\partial x}, Y_3 = \frac{\partial}{\partial y}, Y_4 = \frac{\partial}{\partial z}, Y_5 = \frac{\partial}{\partial u}. \tag{3.22}$$

Now, ES of equation (3.2), $Y = Y_1 + aY_2 + \beta Y_3 + \gamma Y_4, + \psi Y_5$. The operator's for similarity variables are

$$Y = \frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z} + \psi\frac{\partial}{\partial u}. \tag{3.23}$$

Infinitesimal operators include

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial z} + \eta^1\frac{\partial}{\partial y}, \tag{3.24}$$

where, $\xi^1 = 1$, $\xi^2 = a$, $\xi^3 = \beta$, $\xi^4 = \gamma$, $\eta^1 = \psi$. The change of coordinates to canonical form
$v(r,s,w) + \psi t = u(t,x,y,z), w = -at + x, r = -\beta t + y, s = -\gamma t + z, i = t.$
(3.25) Calculating the required partial derivative;s from equation (3.25). We get

$$ux = vw, uxx = vww, uy = vr, uyy = vrr, uz = vs, uzz = vss\,, \tag{3.26}$$

$u_t = -av_w - \gamma v_s - \beta v_r + \psi, u_{tt} = a^2 v_{ww} + 2a\gamma v_{sw} +$
(3.27)

$2a\beta v_{rw} + 2\beta\gamma v_{rs} + \gamma\ v_{ss} + \beta\ v_{rr}\,,$
$u^2{}_t = \psi^2 - 2a\psi v_w + a^2 v_w - 2\psi\gamma v_s + 2a\gamma v_w v_s + \gamma^2 v_s -$
(3.28) 2
$2\psi\beta v_r + 2a\beta v_w v_r + 2\beta\gamma v_s v_r + \beta\ v_{rr}\,.$
Simplifying equation using the values from (3.25) to (3.28) and values of f_u and f_{uu} in equation (3.20).

$$\begin{aligned} &\frac{-\hbar^2}{2m}[v_{ww} + v_{rr} + v_{ss}] + \phi(v + \psi t) - i\hbar[-\alpha v_w - \gamma v_s - \beta v_r + \psi] - \\ &(a(v+\psi t)\ln(1+\tan(v+\psi t)) + \frac{a}{2}\ln(tan^2(v+\psi t)) + b(v+ \\ &\psi t)))[\alpha^2 v_{ww} + 2\alpha\gamma v_{sw} + 2\alpha\beta v_{rw} + 2\beta v_{rs}\gamma + \gamma^2 v_{ss} + \beta^2 v_{rr}] - \\ &(a\ln(1+\tan(v+\psi t)) + 2a\ln(tan(v+\psi t)) + b)[\psi^2 - 2\alpha\psi v_w + \\ &\alpha^2 v_w - 2\psi\gamma v_s + 2\alpha\gamma v_w v_s + \gamma^2 v_s - 2\psi\beta v_r + 2\alpha\beta v_w v_r + \\ &2\beta\gamma v_s v_r + \beta^2 v_{rr}] = 0\,. \end{aligned} \tag{3.29}$$

Simplifying equation using $m = j, i\hbar = k$ in equation (3.29).

$$\begin{aligned} &\frac{-\hbar^2}{2j}[v_{ww} + v_{rr} + v_{ss}] + \phi(v + \psi t) - k[-\alpha v_w - \gamma v_s - \beta v_r + \psi] - \\ &(a(v+\psi t)\ln(1+\tan(v+\psi t)) + \frac{a}{2}\ln(tan^2(v+\psi t)) + b(v+ \\ &\psi t)))[\alpha^2 v_{ww} + 2\alpha\gamma v_{sw} + 2\alpha\beta v_{rw} + 2\beta v_{rs}\gamma + \gamma^2 v_{ss} + \beta^2 v_{rr}] - \\ &(a\ln(1+\tan(v+\psi t)) + 2a\ln(tan(v+\psi t)) + b)[\psi^2 - 2\alpha\psi v_w + \\ &\alpha^2 v_w - 2\psi\gamma v_s + 2\alpha\gamma v_w v_s + \gamma^2 v_s - 2\psi\beta v_r + 2\alpha\beta v_w v_r + \\ &2\beta\gamma v_s v_r + \beta^2 v_{rr}] = 0,. \end{aligned} \tag{3.30}$$

Solving the equation (3.30) which give four LPS.

$$Y_1 = \frac{\partial}{\partial v}, Y_2 = \frac{\partial}{\partial r}, Y_3 = \frac{\partial}{\partial s}, Y_4 = \frac{\partial}{\partial w}. \tag{3.31}$$

Infinitesimal operators include

$$Y = \xi^1 \frac{\partial}{\partial v} + \xi^2 \frac{\partial}{\partial r} + \xi^3 \frac{\partial}{\partial s} + \eta^1 \frac{\partial}{\partial w}, \qquad (3.32)$$

where, $\xi^3 = \beta$, $\xi^2 = a$, , $\xi^1 = 1$, $\eta^1 = \psi$. The coordinates transformation in canonical form:

$$q(m,n) + \psi r = v(r,s,w), m = -br + w, n = -ar + s, p = r . \qquad (3.33)$$

Finding the relevant partial derivatives from the equation (3.33). We get

$$vr = \psi - aqn - bqm, vrr = a2qnn + 2abqmn + b2qmm, vrs = -aqnn - bqmn ,$$
(3.34)

$$vs = qn, vss = qnn, vw = qm, vww = qmm, vsw = qmn, vrw = -aqmn - qmm ,$$
(3.35)

$$vsvw = qmqn, vwvr = \psi qm - aqmqn - bqmqm , \qquad (3.36)$$

$$v_r v_s = \psi q_n - aq_n q_n - bq_m q_n . \qquad (3.37)$$

Reducing the equation with the values derived from the previous equations (3.33) to (3.37) in equation (3.30).

$$\begin{aligned} &\frac{-\hbar^2}{2j}[q_{mm} + a^2 q_{nn} + 2ab q_{mn} + b^2 q_{mm} + q_{nn}] + \phi(q + \psi t + \psi r) - \\ &k[-\alpha q_m - \gamma q_n - \beta\psi + a\beta q_n + b\beta q_m + \psi] - (a(q + \psi t + \psi r)\ln(1 + \\ &\tan(q + \psi t + \psi r)) + \frac{a}{2}\ln(tan^2(q + \psi t + \psi r)) + b(q + \psi t + \\ &\psi r))[\alpha^2 q_{mm} + 2\alpha\gamma q_{mn} - 2\alpha a\beta q_{mn} - 2\alpha\beta b q_{mm} - 2\alpha\beta\gamma a q_{nn} - \\ &2\alpha\beta\gamma b q_{mn} + \gamma^2 q_{nn} + a^2\beta^2 q_{nn} + 2ab\beta^2 q_{mn} + b^2\beta^2 q_{mm}] - (a\ln(1 + \\ &\tan(q + \psi t + \psi r)) + 2a\ln(tan(q + \psi t + \psi r)) + b)[\psi^2 - 2\alpha\psi q_m + \\ &\alpha^2 q_m - 2\alpha\psi q_n + 2\alpha\gamma q_n q_m + \gamma^2 q_n - 2\psi^2\beta + 2a\psi\beta q_n + 2b\psi\beta q_m + \\ &2\alpha\beta\psi q_m - 2a\alpha\beta q_m q_n - 2b\alpha\beta q_m q_m + 2\beta\gamma\psi q_n - 2\alpha\beta\gamma q_n q_n - \\ &2\alpha\beta^2\gamma q_m q_n + a^2\beta^2 q_{nn} + 2ab\beta^2 q_{mn} + b^2\beta^2 q_{mm}] = 0 . \end{aligned} \qquad (3.38)$$

Solving equation (3.38) which give three Lie point symmetries.

$$Y_1 = \frac{\partial}{\partial q}, Y_2 = \frac{\partial}{\partial m}, Y_3 = \frac{\partial}{\partial n}. \qquad (3.39)$$

The infinitesimal operator is

$$Y = \xi^1 \frac{\partial}{\partial q} + \xi^2 \frac{\partial}{\partial m} + \eta^1 \frac{\partial}{\partial n}, \qquad (3.40)$$

where, $\eta^1 = \phi$, $\xi^1 = 1$, $\xi^2 = a$. The coordinates transformation in canonical form:

$$q(m,n) = p(l) + vm, l = -dm + n, o = m. \qquad (3.41)$$

Calculating the required partial derivative;s from equation (3.41).

$$qm = v - dpl, qmm = d2pll, qmqn = vpl - dp2l ,$$
(3.42)

$$qn = pl, qnn = pll, qmn = -dpll .$$

These transformations which are used to diminish the modied Schro¨dinger equation to ODE in the following steps. In the first step equation is diminished to one dimentional. Using the values from the equation (3.41) to (3.42) in equation (3.38), then the simplify equation is

$$\begin{aligned} &\frac{-\hbar^2}{2j}[d^2 p_{ll} + a^2 p_{ll} - 2abd p_{ll} + b^2 d^2 p_{ll} + p_{ll}] + \phi(p(l) + \psi i + \\ &\psi p + \nu o) - k[-\alpha\nu + \alpha d p_l - \gamma p_l - \beta\psi + a\beta p_l + b\beta\nu - b\beta d p_l + \psi] - \\ &(a(p(l) + \psi i + \psi p + \nu o)\ln(1 + \tan(p(l) + \psi i + \psi p + \nu o)) + \\ &\frac{a}{2}\ln(tan^2(p(l) + \psi i + \psi p + \nu o)) + b(p + \psi i + \psi p + \nu o)))[\alpha^2 d^2 p_{ll} - \\ &2\alpha\gamma d p_{ll} + 2\alpha a\beta d p_{ll} - 2\alpha\beta b d^2 p_{ll} - 2\alpha\beta\gamma a p_{ll} + 2\alpha\beta\gamma b d p_{ll} + \gamma^2 p_{ll} + \\ &a^2\beta^2 p_{ll} - 2ab\beta^2 d p_{ll} + b^2\beta^2 d^2 p_{ll}] - (a\ln(1 + \tan(p(l) + \psi i + \psi p + \\ &\nu o)) + 2a\ln(tan(p(l) + \psi i + \psi p + \nu o)) + b)[\psi^2 - 2\alpha\psi\nu + 2\alpha\psi d p_l + \\ &\alpha^2\nu - \alpha^2 d p_l - 2\alpha\psi p_l + 2\alpha\gamma\nu p_l - 2\alpha\gamma d p_l^2 + \gamma^2 p_l - 2\psi^2\beta + \\ &2\alpha\psi\beta p_l + 2b\psi\beta\nu - 2bd\beta\psi p_l + 2\alpha\beta\psi\nu - 2\alpha\beta\psi d p_l - 2a\alpha\beta\nu p_l + \\ &2a\alpha\beta d p_l^2 - 2b\alpha\beta\nu p_l + 2b\alpha\beta d p_l^2 + 2b\beta\gamma\psi p_l - 2\alpha\beta p_l^2 - \\ &2\alpha\beta^2\gamma\eta p_l + 2\alpha\beta^2\gamma d p_l^2 + a^2\beta^2 p_{ll} - 2abd\beta^2 p_{ll} + b^2 d^2\beta^2 p_{ll}] = 0 . \end{aligned} \qquad (3.43)$$

Now, we will examine different cases of solutions to the equation (3.43).
Case 1: When $a = 0,\beta = 0,\gamma = 0,i = 0,\psi = 0,\nu = 0,p = 0,o = 0,a = 0,b = 0,c = 0,d = 0$,

$$p(l) = C_1 e^{\frac{\sqrt{2\phi j}l}{\hbar}} + C_2 e^{-\frac{\sqrt{2\phi j}l}{\hbar}} \quad (3.44)$$

Now, solution takes $u(t,x,y,z)$

$$u(t,x,y,z) = C_1 e^{\frac{\sqrt{2\phi j}z}{\hbar}} + C_2 e^{-\frac{\sqrt{2\phi j}z}{\hbar}}, \quad (3.45)$$

is a solution of (3.20).

Case 2: When $a = 0,b = 0,c = 0,d = 0,\gamma = 0,a = 0,\beta = 0$,

$$p(l) = C_2 e^{\frac{\sqrt{2\phi j}l}{\hbar}} + C_1 e^{-\frac{\sqrt{2\phi j}l}{\hbar}} + \frac{((-i-p)\phi - \nu o)\phi + k\psi}{\phi}. \quad (3.46)$$

Now, solution takes $u(t,x,y,z)$

$$u(t,x,y,z) = C_2 e^{\frac{\sqrt{2\phi j}z}{\hbar}} + C_1 e^{-\frac{\sqrt{2\phi j}z}{\hbar}} + \frac{((-t-y)\phi - \nu x)\phi + k\psi}{\phi} + \nu x + \psi y + \psi t, \quad (3.47)$$

is ES of Eq. (3.20).

Case 3: When $a = 0$, $a = 0,b = 0,\beta = 0,c = 0,d = 0$,
(3.48)

$$p(l) = C_2 e^{\frac{(kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})l}{\hbar^2}} + C_1 e^{-\frac{(-kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})l}{\hbar^2}} + \frac{((-i-p)\psi - \nu o)\phi + k\psi}{\phi}.$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z) = C_2 e^{\frac{(kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})z}{\hbar^2}} + C_1 e^{-\frac{(-kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})z}{\hbar^2}} + \frac{((-t-y)\psi - \nu x)\phi + k\psi}{\phi} + \nu x + \psi y + \psi t, \quad (3.49)$$

satisfies equation (3.20).
Case 4: When $a = 0,c = 0,d = 0,b = 0$
(3.50)

$$p(l) = C_2 e^{\frac{(kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})l}{\hbar^2}} + C_1 e^{-\frac{(-kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})l}{\hbar^2}} + \frac{((-t-(-\beta t + y))\psi - \nu(-\alpha t + x)\phi - k(\psi(\beta - 1) + \alpha\nu)}{\phi}.$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z) = C_2 e^{\frac{(kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})(\gamma t + z)}{\hbar^2}} + C_1 e^{-\frac{(-kj\gamma + \sqrt{k^2 j^2 \gamma^2 + 2\phi j \hbar^2})(\gamma t + z)}{\hbar^2}} + \frac{((-i-p)\psi - \nu o)\phi - k(\psi(\beta - 1) + \alpha\nu)}{\phi} + \nu(-\alpha t + x) + \psi(-\beta t + y) + \psi t, \quad (3.51)$$

is a solution of (3.20).

Case 5: When $a = 0,a = 0,b = 0,c = 0,d = 0,\gamma = 0$

$$p(l) = C_2 e^{\frac{\sqrt{2\phi j}l}{\hbar}} + C_1 e^{-\frac{\sqrt{2\phi j}l}{\hbar}} + \frac{((-i-p)\phi - \nu o)\phi - k\psi(\beta - 1)}{\phi}. \quad (3.52)$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z) = C_2 e^{\frac{\sqrt{2\phi j}z}{\hbar}} + C_1 e^{-\frac{\sqrt{2\phi j}z}{\hbar}} + \frac{((-i-p)\phi - \nu o)\phi - k\psi(\beta - 1)}{\phi} + \nu x + \psi(-\beta t + y) + \psi t,$$

(3.53)is a solution of (3.20).

The classical Lie symmetry
transformations are calculated by the following way, using equation (3.17) after taking the first and second derivative's.

$$f_u = \frac{ae^{bu}}{b} + \frac{cu^2}{2} + du$$

$$f_{uu} = ae^{bu} + cu + d \quad (3.54)$$

(3.55)

3.3GMSELie Symmetries

Involving $f(u) = \frac{ae^{bu}}{b^2} + \frac{cu^3}{6} + \frac{du^2}{2}$

If $f_u = \frac{ae^bu}{b} + \frac{cu^2}{2} + du$ and $f_{uu} = ae^{bu} + cu + d$ then equation (3.2) becomes

(3.56)

$$\frac{-\hbar^2}{2m}[u_{xx} + u_{yy} + u_{zz}] + \phi u - i\hbar u_t - (\frac{ae^{bu}}{b} + \frac{cu^2}{2} + du)u_{tt} - (ae^{bu} + cu + d)u_t^2 = 0 .$$

Solving the equation (3.56) which give to five LPS

$$Y_1 = \frac{\partial}{\partial t}, Y_2 = \frac{\partial}{\partial x}, Y_3 = \frac{\partial}{\partial y}, Y_4 = \frac{\partial}{\partial z}, Y_5 = \frac{\partial}{\partial u}. \quad (3.57)$$

3.4GMSE Exact Solutions Involving $f(u) = \frac{ae^{bu}}{b^2} + \frac{cu^3}{6} + \frac{du^2}{2}$

Next according to the above five Lie point symmetrieswe we find ES of equation (3.56)

$$Y_1 = \frac{\partial}{\partial t}, Y_2 = \frac{\partial}{\partial x}, Y_3 = \frac{\partial}{\partial y}, Y_4 = \frac{\partial}{\partial z}, Y_5 = \frac{\partial}{\partial u}. \quad (3.58)$$

Now for ES of equation (3.56) using $Y = Y_1 + aY_2 + \beta Y_3 + \gamma Y_4, + \psi Y_5$. The analogy variables for operator are

$$Y = \frac{\partial}{\partial t} + \alpha\frac{\partial}{\partial x} + \beta\frac{\partial}{\partial y} + \gamma\frac{\partial}{\partial z} + \psi\frac{\partial}{\partial u}. \quad (3.59)$$

The infinitesimal operator include

$$Y = \xi^1\frac{\partial}{\partial t} + \xi^2\frac{\partial}{\partial x} + \xi^3\frac{\partial}{\partial y} + \xi^3\frac{\partial}{\partial z} + \eta^1\frac{\partial}{\partial u}, \quad (3.60)$$

where, $\xi^1 = 1$, $\xi^2 = a$, $\xi^3 = \beta, \xi^4 = \gamma$, $\eta^1 = \psi$. The change in coordinates of canonical form:

$r = v(r,s,w) + \psi t = u(t,x,y,z), -\beta t + y, s = -\gamma t + z, w = -at + x, i = t.$

(3.61) Finding the relevant partial derivatives from the equation (3.61). We get

$ux = vw, uxx = vww, uy = vr, uyy = vrr, uz = vs, uzz = vss$, (3.62)

$u_t = -av_w - \gamma v_s - \beta v_r + \psi$, (3.63)

$utt = a2vww + 2a\gamma vsw + 2a\beta vrw + 2\beta\gamma vrs + \gamma 2vss + \beta 2vrr$. (3.64)

Finding value of $u^2{}_t$ from value of u_t.

$u^2{}_t = \psi^2 - 2a\psi v_w + a^2 v_w - 2\psi\gamma v_s + 2a\gamma v_w v_s + \gamma^2 v_s - 2\psi\beta v_r +$

2 (3.65)

$2a\beta v_w v_r + 2\beta\gamma v_s v_r + \beta\ v_{rr}$.

Simplifying equation using the values from (3.61) to (3.65) and values of f_u and f_{uu} in equation (3.56).

$$\frac{-\hbar^2}{2m}[v_{ww} + v_{rr} + v_{ss}] + \phi(v + \psi t) - \iota\hbar[-\alpha v_w - \gamma v_s - \beta v_r + \psi] - (\frac{ae^{b(v+\psi t)}}{b} + \frac{c(v+\psi t)^2}{2} + dv)[\alpha^2 v_{ww} + 2\alpha\gamma v_{sw} + 2\alpha\beta v_{rw} + 2\beta v_{rs}\gamma + \gamma^2 v_{ss} + \beta^2 v_{rr}] - ((ae^{b(v+\psi t)} + c(v+\psi t) + d))[\psi^2 - 2\alpha\psi v_w + \alpha^2 v_w - 2\psi\gamma v_s + 2\alpha\gamma v_w v_s + \gamma^2 v_s - 2\psi\beta v_r + 2\alpha\beta v_w v_r + 2\beta\gamma v_s v_r + \beta^2 v_{rr}] = 0. \quad (3.66)$$

Using $m = f, i\hbar = k$ in equation (3.66).

$$\frac{-\hbar^2}{2f}[v_{ww}+v_{rr}+v_{ss}]+\phi(v+\psi t)-k[-\alpha v_w-\gamma v_s-\beta v_r+\psi]- (\frac{ae^{b(v+\psi t)}}{b}+\frac{c(v+\psi t)^2}{2}+dv)[\alpha^2 v_{ww}+2\alpha\gamma v_{sw}+2\alpha\beta v_{rw}+ 2\beta v_{rs}\gamma+\gamma^2 v_{ss}+\beta^2 v_{rr}]-((ae^{b(v+\psi t)}+c(v+\psi t)+d))[\psi^2- 2\alpha\psi v_w+\alpha^2 v_w-2\psi\gamma v_s+2\alpha\gamma v_w v_s+\gamma^2 v_s-2\psi\beta v_r+2\alpha\beta v_w v_r+ 2\beta\gamma v_s v_r+\beta^2 v_{rr}]=0. \quad (3.67)$$

Solving the equation (3.67) which give fourLPS.

$$Y_1=\frac{\partial}{\partial v},Y_2=\frac{\partial}{\partial r},Y_3=\frac{\partial}{\partial s},Y_4=\frac{\partial}{\partial w}. \quad (3.68)$$

The infinitesimal operator is

$$Y=\xi^1\frac{\partial}{\partial v}+\xi^2\frac{\partial}{\partial r}+\xi^3\frac{\partial}{\partial s}+\eta^1\frac{\partial}{\partial w}, \quad (3.69)$$

where, $\eta^1=\psi$, $\xi^1=1$, $\xi^2=a$, $\xi^3=\beta$. The change of coordinates to canonical form:

$v(r,s,w) = q(m,n) + \psi r, m = -br + w, n = -ar + s, p = r$. (3.70)

Finding the relevant partial derivatives from the equation (3.70). We get

$vr = \psi - aqn - bqm, vrr = a2qnn + 2abqmn + b2qmm, vs = qn, vss = qnn$,
(3.71)

$vw = qm, vww = qmm, vsw = qmn, vrw = -aqmn - qmm, vrs = -aqnn - bqmn$,
(3.72)

$vsvw = qmqn, vwvr = \psi qm - aqmqn - bqmqm$, (3.73)

$v_r v_s = \psi q_n - aq_n q_n - bq_m q_n$. (3.74)

Reducing the equation with the values derived from the previous equations (3.71) to (3.74) in equation (3.67).

$$\frac{-\hbar^2}{2f}[q_{mm}+a^2q_{nn}+2abq_{mn}+b^2q_{mm}+q_{nn}]+\phi(q+\psi t+\psi r)- k[-\alpha q_m-\gamma q_n-\beta\psi+a\beta q_n+b\beta q_m+\psi]-((\frac{ae^{b((q+\psi t+\psi r)}}{b}+ \frac{c((q+\psi t+\psi r)^2}{2}+d(q+\psi t+\psi r))[\alpha^2 q_{mm}+2\alpha\gamma q_{mn}-2\alpha a\beta q_{mn}- 2\alpha\beta b q_{mm}-2\alpha\beta\gamma a q_{nn}-2\alpha\beta\gamma b q_{mn}+\gamma^2 q_{nn}+a^2\beta^2 q_{nn}+2ab\beta^2 q_{mn}+ b^2\beta^2 q_{mm}]-(ae^{b((q+\psi t+\psi r)}+c((q+\psi t+\psi r)+d)[\psi^2-2\alpha\psi q_m+\alpha^2 q_m- 2\alpha\psi q_n+2\alpha\gamma q_n q_m+\gamma^2 q_n-2\psi^2\beta+2a\psi\beta q_n+2b\psi\beta q_m+2\alpha\beta\psi q_m- 2a\alpha\beta q_m q_n-2b\alpha\beta q_m q_m+2\beta\gamma\psi q_n-2\alpha\beta\gamma q_n q_n-2\alpha\beta^2\gamma q_m q_n+ a^2\beta^2 q_{nn}+2ab\beta^2 q_{mn}+b^2\beta^2 q_{mm}]=0. \quad (3.75)$$

Solving the above determining equations give to three LPS

$$Y_1=\frac{\partial}{\partial q},Y_2=\frac{\partial}{\partial m},Y_3=\frac{\partial}{\partial n}. \quad (3.76)$$

Operator function in infinitesimal form:

$$Y=\xi^1\frac{\partial}{\partial q}+\xi^2\frac{\partial}{\partial m}+\eta^1\frac{\partial}{\partial n}, \quad (3.77)$$

where, $\eta^1=\phi,\xi^1=1,\xi^2=a$.

The change in coordinates of canonical form:

$q(m,n) = p(l) + \eta m, l = -dm + n, o = m.$ (3.78)

Calculating the required partial derivative;s from equation (3.78).

$qm = \eta - dpl, qmm = d2pll, qn = pl, qnn = pll, qmqn = \eta pl - dp2l, qmn = -dpll$.

(3.79) These transformations are used to diminish the GMS equation to ordinary differential equation in the following steps. In the

1st step equation is diminished to one dimensional.

Reducing the equation with the values derived from the previous equations (3.78) to (3.79) in equation (3.75).

$$\frac{-\hbar^2}{2f}[d^2p_{ll}+a^2p_{ll}-2abdp_{ll}+b^2d^2p_{ll}+p_{ll}]+\phi(p(l)+\eta o+\psi i+\psi p)-k[-\alpha\eta+\alpha dp_l-\gamma p_l-\beta\psi+a\beta p_l+b\beta\eta-b\beta dp_l+\psi]-(\frac{ae^{b(((p(l)+\eta o+\psi i+\psi p)}}{b}+\frac{c(((p(l)+\eta o+\psi i+\psi p)^2}{2}+d((p(l)+\eta o+\psi i+\psi p)[\alpha^2d^2p_{ll}-2\alpha\gamma dp_{ll}+2\alpha a\beta dp_{ll}-2\alpha\beta bd^2p_{ll}-2\alpha\beta\gamma ap_{ll}+2\alpha\beta\gamma bdp_{ll}+\gamma^2p_{ll}+a^2\beta^2p_{ll}-2ab\beta^2dp_{ll}+b^2\beta^2d^2p_{ll}]-(ae^{b((p(l)+\eta o+\psi i+\psi p)}+c(p(l)+\eta o+\psi i+\psi p)+d)[\psi^2-2\alpha\psi\eta+2\alpha\psi dp_l+\alpha^2\eta-\alpha^2dp_l-2\alpha\psi p_l+2\alpha\gamma\eta p_l-2\alpha\gamma dp_l^2+\gamma^2p_l-2\psi^2\beta+2\alpha\psi\beta p_l+2b\psi\beta\eta-2bd\beta\psi p_l+2\alpha\beta\psi\eta-2\alpha\beta\psi dp_l-2a\alpha\beta\eta p_l+2a\alpha\beta dp_l^2-2b\alpha\beta\eta p_l+2b\alpha\beta dp_l^2+2b\beta\gamma\psi p_l-2\alpha\beta p_l^2-2\alpha\beta^2\gamma\eta p_l+2\alpha\beta^2\gamma dp_l^2+a^2\beta^2p_{ll}-2abd\beta^2p_{ll}+b^2d^2\beta^2p_{ll}]=0\,. \tag{3.80}$$

Now, we will examine different cases of solutions to the equation (3.80).

Case 1: When $a=0,\beta=0,\gamma=0,\eta=0,i=0,\psi=0,p=0,o=0$

$$p(l)=C_1e^{\frac{\sqrt{2\phi f}l}{\hbar\sqrt{d^2+a^2-2abd+b^2d^2+1}}}+C_2e^{-\frac{\sqrt{2\phi f}l}{\hbar\sqrt{d^2+a^2-2abd+b^2d^2+1}}}. \tag{3.81}$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z)=C_1e^{\frac{\sqrt{2\phi f}-d(-by+x)+(-ay+z)}{\hbar\sqrt{d^2+a^2-2abd+b^2d^2+1}}}+C_2e^{-\frac{\sqrt{2\phi f}-d(-by+x)+(-ay+z)}{\hbar\sqrt{d^2+a^2-2abd+b^2d^2+1}}}, \tag{3.82}$$

is ES of Eq. (3.56).

Case 2: When $\eta=0,a=0,\beta=0,a=0,b=0,c=0,d=0,\gamma=0,$

$$p(l)=C_2e^{\frac{\sqrt{2\phi f}l}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}l}{\hbar}}-\frac{\psi((i+p)\phi-k)}{\phi}. \tag{3.83}$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z)=C_2e^{\frac{\sqrt{2\phi f}z}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}z}{\hbar}}-\frac{\psi((t+y)\phi-k)}{\phi}+\psi y+\psi t\,, \tag{3.84}$$

is ES of Eq(3.56).

Case 3: When $a=0,\beta=0,a=0,b=0,c=0,d=0,\gamma=0,\psi=0,$

$$p(l)=C_2e^{\frac{\sqrt{2\phi f}l}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}l}{\hbar}}-\eta o\,. \tag{3.85}$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z)=C_2e^{\frac{\sqrt{2\phi f}z}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}z}{\hbar}}, \tag{3.86}$$

is ES of Eq(3.56).

Case 4: When $a=0,b=0,c=0,d=0,\gamma=0,a=0,\beta=0,$

$$p(l)=C_2e^{\frac{\sqrt{2\phi f}l}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}l}{\hbar}}+\frac{((-i-p)\psi-\eta o)\phi+k\psi}{\phi}. \tag{3.87}$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z)=C_2e^{\frac{\sqrt{2\phi f}z}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}z}{\hbar}}+\frac{((-t-y)\psi-\eta x)\phi+k\psi}{\phi}+\eta x+\psi y+\psi t\,, \tag{3.88}$$

is ES of (3.56).

Case 5: When $a=0,i=0,a=0,b=0,c=0,d=0,\beta=0,\gamma=0,$

$$p(l)=C_2e^{\frac{\sqrt{2\phi f}l}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}l}{\hbar}}+\frac{-\phi\psi p-\phi\eta o+k\psi}{\phi}. \tag{3.89}$$

Now, solution takes $u(t,x,y,z)$ shape

$$u(t,x,y,z)=C_2e^{\frac{\sqrt{2\phi f}z}{\hbar}}+C_1e^{-\frac{\sqrt{2\phi f}z}{\hbar}}+\frac{-\phi\psi y-\phi\eta x+k\psi}{\phi}+\eta x+\psi y+\psi t\,, \tag{3.90}$$

is a solution of (3.56).

3.5Discussion

In this section we explores the invariant solutions of the GMS equation using the Lie method. We start by applying the second-order prolongation formula to determine the unknown functions involved in the equation. The application of the second-order prolongation formula to the GMS equation is presented in Section 2. Afterward, we derive the Lie point symmetries of the equation to incorporate the values of these functions. Using invariant transformations, we reduce the given PDEs to ODEs and find their exact solutions.

4 Conclusion

Through the Lie approach, we found various solutions for the GMS equation and determined its Lie point symmetry group. Applying symmetry reduction three times on the given GMS PDEs results in their transformation into ordinary differential equations (ODEs). By applying symmetry reduction three times to the given GMS PDEs, we convert them into ordinary differential equations (ODEs) and construct their invariant solutions. Closed form exact solutions for the GMS equation were achieved through the combination of constructed Lie symmetries. It is important to note that the symmetry approach used to obtain closed-form solutions for the given equation has not been reported in previous literature. Consequently, the findings in this work are new and serve as an extension of the research on the GMS equation, as discussed in the article.

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