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An Improved Exponential Estimator Of The Finite Population Mean In Simple Random Sampling Utilizing Double Auxiliary Variables

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Abstract
This study introduces a novel family of estimators for the finite population mean that demonstrate improved accuracy by utilizing dual auxiliary information in simple random sampling. The proposed estimators were evaluated by deriving their bias and Mean Square Error (MSE) expressions up to the first-order approximation. The analysis identifies the conditions under which these estimators outperform existing methods. Real-world data were used to compute the MSE and Percentage Relative Efficiency (PRE) of the proposed estimators. Comparative results show that, under specified conditions, the new estimator family achieve greater precision, reducing MSE and enhancing estimation accuracy.

Key Words: Auxiliary; Bias; Efficiency; MSE; PRE; Simple Random Sampling;

Introduction

In survey sampling, the effectiveness of an estimator for a population parameter is often enhanced by incorporating auxiliary information, especially when this information is closely related to the study variables. Auxiliary data play a crucial role in selecting and estimating population parameters, leading to more precise estimates of unknown population parameters. Generally, the efficiency of these estimates improves as the number of auxiliary variables increases. Well-known estimation methods such as ratio, product, and difference methods are commonly used in survey sampling. Ratio-type estimations are particularly effective when there is a strong positive correlation between the study and auxiliary variables. Conversely, product-type estimations are useful when there is a strong negative correlation. Various researchers have also explored regression-type and exponential-type estimators based on different transformations. The primary goal of this research is to develop a new estimator that can predict the population mean more accurately than existing estimators.

When the population of auxiliary variables is known beforehand, using various regression-type, ratio-type, and product-type estimators is widely accepted in survey sampling literature for estimating the population mean of a study variable. Several researchers have put forth novel estimators, claiming enhanced efficiency compared to established alternatives. For instance, [1] developed and evaluated an efficient estimator for estimating the population mean.

Survey sampling often utilizes supplementary data to enhance the accuracy of estimations. This approach was initially introduced by [2,3], who incorporated additional information into ratio and regression estimation techniques. In recent years, various researchers have proposed different types of ratio estimators by effectively transforming auxiliary variables. To delve deeper into these research developments, readers can refer to the works of [4,5,6,7,8,9,10] and the references cited in these studies. When the correlation is negative in product-type estimators, other researchers have proposed alternative estimators in various literature. The use of auxiliary information in the form of exponential ratio and product-type estimators was first introduced by [11]. Since then, numerous other studies have explored the use of auxiliary variables in this context, including works by [12,13,14,15,16,17,18,19 and 20].

For future studies, refer to the additional literature on the suggested ratio-cum-product estimator for the population mean cited in [21,22,23,24,25,26.27, and 28]. Furthermore, building on foundational work, this study introduces an innovative family of exponential ratio-cum-product estimators designed to enhance the precision of parameter estimation in survey sampling. The objectives are dual: first, to develop these estimators for more accurate population mean estimation, and second, to compare their performance with existing methods, demonstrating their effectiveness under various sampling conditions. For futher studies see the [29,30,31, and 32].

Introducing new, precise families of population mean estimators that integrate both ratio and product estimators. These estimators also incorporate additional population parameters, such as correlation coefficients and coefficients of variation from auxiliary variables, to further enhance their precision.

Methodology

Consider a population denoted as $(\Psi = \psi_1, \psi_2, \ldots . \psi_N)$ with a size of N unites to be obtained a random sample of n sample unites to be selected by using simple random sampling scheme (in simple random sampling each units of the population to be selected has an equal chance of being included in the sample) without replacement method. In simple random sampling let suppose we have the study variables sample information "y" to estimate the population mean. On the other hand, let us

consider two auxiliary variables, "X_1" and "X_2." To support this estimation, we have data available from these two additional variables. It is significant to highlight that information about the variables and the primary study variable is easily available.
Let assumes that $\varepsilon_0 = \frac{1}{\bar{Y}}(\bar{y} - \bar{Y}), \varepsilon_1 = \frac{1}{\bar{X}}(\bar{x}_1 - \bar{X}_1)$ and $\varepsilon_2 = \frac{1}{\bar{Z}}(\bar{x}_2 - \bar{X}_2)$ It fulfills the following ensuing characteristics.
$E(\varepsilon_0) = E(\varepsilon_1) = E(\varepsilon_2) = 0$, $E(\varepsilon_0^2) = \lambda C_y^2$, $E(\varepsilon_1^2) = \lambda C_{\bar{x}_1}^2$, $E(\varepsilon_2^2) = \lambda C_{\bar{x}_2}^2$, $E(\varepsilon_0\ \varepsilon_1) = \lambda C_{y\bar{x}_1}$, $E(\varepsilon_0\ \varepsilon_2) = \lambda C_{y\bar{x}_2}$, and $E(\varepsilon_1\ \varepsilon_2) = \lambda C_{\bar{x}_1\bar{x}_2}$, in addition, where $\lambda = \frac{N-n}{nN}$
Furthermore, $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} yi,\ \bar{X}_1 = \frac{1}{N}\sum_{i=1}^{N} xi,\ and\ \bar{X}_2 = \frac{1}{N}\sum_{i=1}^{N} zi,\ S_y^2 = \frac{1}{N}\sum_{i=1}^{N}(y - \bar{Y})^2,\ S_x^2 = \frac{1}{N}\sum_{i=1}^{N}(x_1 - \bar{X}_1)^2,\ S_x^2 = \frac{1}{N}\sum_{i=1}^{N}(x_2 - \bar{X}_2)^2,\ C_{yx_1} = \rho_{yx_1\ C_yC_{x_1}},\ C_{yx_2} = \rho_{yx_2\ C_yC_{x_2}},\ C_{x_1x_2} = \rho_{x_1x_2\ C_{x_1}C_{x_2}}$
The incorporation of supplemental data not only helps to lessen bias but also lowers estimation variability, which improves performance overall. Consequently, by utilizing the power of additional information throughout the estimation process, the dual auxiliary approach offers a reliable way to improve estimator efficiency.

Literature-based estimators

The following estimators are taken into consideration in this section. In the discipline of statistics, many estimators for determining the population mean have been created and recorded. To ensure the most appropriate and precise estimation of the population mean, researchers in the literature frequently select from this pool of estimators based on the unique features of their data and the underlying assumptions that support their research goals.
When the data only includes the research variable, the conventional, traditional estimate of mean $T_1 = \bar{y}$ is employed. And their estimator's variance is given in Eq. (1)

$$Var(T_1) = \bar{Y}^2\lambda C_y^2 \quad (1)$$

As according to [2], the ratio estimator in the context of utilizing dual auxiliary variables, which is given in Eq. (2) as follows:

$$T_2 = \bar{y}\left(\frac{\bar{X}_1}{\bar{x}_1}\right)\left(\frac{\bar{X}_2}{\bar{x}_2}\right) \quad (2)$$

The Mean Squared Error (MSE) of the aforementioned ratio estimator is provided in Eq. (3) as follows.

$$\text{MSE}\ (T_2) = \lambda\bar{Y}^2C_y^2 + C_{\bar{x}_1}^2 + C_{\bar{x}_2}^2 - 2C_{y\bar{x}_1} - 2C_{y\bar{x}_2} + 2C_{\bar{x}_1\bar{x}_2} \quad (3)$$

The chain ratio-product estimator introduced by [21] is presented in Equation (4) as follows.

$$T_p = \bar{y}\left(\frac{\bar{X}_1}{\bar{x}_1}\right)\left(\frac{\bar{x}_2}{\bar{X}_2}\right) \quad (4)$$

The Mean Squared Error (MSE) of the above estimator, up to the first-order approximation, is demonstrated in Equation (5) as.

$$\text{MSE}\ (T_p) = \lambda\bar{Y}^2\left(C_y^2 + C_{\bar{x}_1}^2 + C_{\bar{x}_2}^2 - 2C_{y\bar{x}_1} + 2C_{y\bar{x}_2} - 2C_{\bar{x}_1\bar{x}_2}\right) \quad (5)$$

The regression estimator using dual auxiliary variables is described in Equation (6). This estimator leverages two auxiliary variables to refine the estimation of the target variable, incorporating their relationships into the regression model.

$$T_{reg} = \bar{y} + b_1\ (\bar{X}_1 - \bar{x}_1) + b_2(\bar{X}_2 - \bar{x}_2) \quad (6)$$

Where $b_1 = \frac{S_y}{S_{\bar{x}_1}^2}$ and $b_2 = \frac{S_{y\bar{x}_2}}{S_{\bar{x}_2}^2}$ are the sample regression coefficients which are associated with population regression coefficients $\beta_1 = \frac{S_{y\bar{x}_1}}{S_{\bar{x}_1}^2}$ and $\beta_2 = \frac{S_{y\bar{x}_2}}{S_{\bar{x}_2}^2}$ respectively.
The MSE of the regression estimator in Equation (7) measures the average squared difference between the model's predicted values and the actual values, accounting for both variance and bias.

$$\text{MSE}\ (T_{reg}) = \lambda\bar{Y}^2C_y^2(1 - \rho_{y\bar{x}_1}^2 - \rho_{y\bar{x}_2}^2 + 2\rho_{y\bar{x}_1}\rho_{y\bar{x}_2}\rho_{\bar{x}_1\bar{x}_2}) \quad (7)$$

The estimator suggested by the author in reference [31] is based on the use of two auxiliary variables and is expressed in exponential format, as shown in Eq (8).

$$T_3 = \left[\delta_3\bar{y} + \delta_4\left(\frac{\bar{X}_1 - \bar{x}_1}{\bar{X}_1}\right) + \delta_5\left(\frac{\bar{X}_2 - \bar{x}_2}{\bar{X}_2}\right)\right] exp\left[\left(\frac{u(\bar{X}_1 - \bar{x}_1)}{u(\bar{X}_1 + \bar{x}_1) + 2v}\right)\right] \quad (8)$$

To find the minimum Mean Squared Error (MSE) of the estimator, we need to determine the values of the constants $\delta_3, \delta_4\ and\ \delta_5$. These constants are adjusted to minimize the resulting MSE. Here, u and v are generalizing constants that depend on various parameters of the auxiliary variables. We define a term $\omega = \frac{u\bar{X}_1}{u\bar{X}_1 + v}$ The expression for the minimum MSE of the estimator is provided in Equation (9).

$$\text{MSE}\ (T_3) = \cong \bar{Y}^2\ \lambda C_y^2(1 - \rho_{yx}^2) - G_1 - G_2 \quad (9)$$

Where $G_1 = \frac{\bar{Y}^2\left(\omega^2C_{\bar{x}_1}^2 - 8C_{y\bar{x}_1}^2 - 8C_y^2C_{\bar{x}_1}^2\right)^2}{64C_{\bar{x}_1}^2\left\{1 + C_y^2\left(1 - \rho_{\bar{x}_1}^2\right)\right\}}\ and\ G_2 = \frac{\bar{Y}^2\left(\omega^2C_{\bar{x}_1}^2 - 8\right)^2\left(C_{\bar{x}_1}^2C_{y\bar{x}_2} - C_{y_{\bar{x}_2}}C_{\bar{x}_1\bar{x}_2}\right)^2}{64C_{\bar{x}_1}^2C_{\bar{x}_2}^2\left(1 - \rho_{\bar{x}_1\bar{x}_2}^2\right)\left\{1 + C_y^2\left(1 - \rho_{\bar{x}_1}^2\right)\right\}\left\{1 + C_y^2\left(1 - \rho_{y\bar{x}_1\bar{x}_2}^2\right)\right\}}$.
In reference [35], a product estimator is proposed, which is expressed in Eq (10).

$$T_4 = \delta_6\bar{y} + \delta_7\bar{y}\left(\frac{\bar{x}_1}{\bar{X}_1}\right)^{\alpha_1} + \delta_8\bar{y}\left(\frac{\bar{x}_2}{\bar{X}_2}\right)^{\alpha_2} \quad (10)$$

In this context, δ_6, δ_7 and δ_8 are constants that need to be determined in a way that minimizes the Mean Squared Error (MSE). The constants $\alpha_1\ and\ \alpha_2$ can take on either positive or negative values. Specifically, $\delta_6 = \frac{\varphi_0}{\varphi}, \delta_7 = \frac{\varphi_1}{\varphi}$ and $\delta_8 = \frac{\varphi_2}{\varphi}$ The minimum MSE for the T_5 estimator is given in Eq (11).

$$\text{MSE}\ (T_4) \cong \bar{Y}^2\left[1 - \frac{\varphi_0 + A_6\varphi_1 + A_7\varphi_2}{\varphi}\right] \quad (11)$$

Where $\varphi = A_0(A_1A_2 - A_5^2) - A_3(A_2A_3 - A_4 A_5) + A_4(A_3A_5 - A_1 A_4), \varphi_0 = (A_1A_2 - A_5^2) - A_3(A_2A_6 - A_5 A_7) + A_4(A_5A_6 - A_1 A_7), \varphi_1 = A_0(A_2A_6 - A_5A_7) - (A_2A_3 - A_4 A_5) + A_4(A_3A_7 - A_4 A_6)$ *and* $\varphi_2 = A_0(A_1A_7 - A_5A_6) - A_3(A_3A_7 - A_4 A_6) + (A_3A_5 - A_1 A_4)$.

Furthermore, $A_0 = 1 + \lambda C_y^2, A_1 = 1 + \lambda\{C_y^2 + 4\alpha_1 C_{y\bar{x}_1} + \alpha_1 (2\alpha_1 - 1)C_{\bar{x}_1}^2\}$, $A_2 = 1 + \lambda\{C_y^2 + 4\alpha_2 C_{y\bar{x}_2} + \alpha_2 (2\alpha_2 - 1)C_{\bar{x}_2}^2\}$, $A_3 = 1 + \lambda\left\{C_y^2 + 2\alpha_1 C_{y\bar{x}_1} + \frac{\alpha_1(\alpha_1-1)}{2}C_{\bar{x}_1}^2\right\}, A_4 = 1 + \lambda\left\{C_y^2 + 2\alpha_2 C_{y\bar{x}_2} + \frac{\alpha_2(\alpha_2-1)}{2}C_{\bar{x}_2}^2\right\}$, $A_5 = 1 + \lambda\left\{C_y^2 + 2\alpha_1 C_{y\bar{x}_1} + 2\alpha_2 C_{y\bar{x}_2} + \alpha_1 \alpha_2 C_{\bar{x}_1\bar{x}_2} \frac{\alpha_1(\alpha_1-1)}{2}C_{\bar{x}_1}^2 \frac{\alpha_2(\alpha_2-1)}{2}C_{\bar{x}_2}^2\right\}$, $A_6 = 1 + \frac{\lambda\alpha_1}{2}\left(\alpha_1 + 2C_{y_{\bar{x}_1}}\left(C_{\bar{x}_1}^2\right)^{-1} - 1\right)C_{\bar{x}_1}^2$ *and* $A_7 = 1 + \frac{\lambda\alpha_2}{2}\left(\alpha_2 + 2C_{y_{\bar{x}_2}}\left(C_{\bar{x}_2}^2\right)^{-1} - 1\right)C_{\bar{x}_2}^2$

Proposed Estimator

This section introduces two groups of proposed estimators for the finite population mean, employing two auxiliary variables within the context of simple random sampling. Additionally, it includes the derivation of formulas for both bias and mean squared error (MSE). The dual exponential introduces by Iftikhar et al [36] presented in the estimate shown below with two auxiliary variables.

Inspired by these studies, a novel family of population mean estimators is proposed in the following Eq. (12) by adjusting [21].

$$T_s = \{k\bar{y} + L(\bar{X}_1 - \bar{x}_1) + M(\bar{X}_2 - \bar{x}_2)\}exp\left\{\frac{a(\bar{X}_1-\bar{x}_1)}{a(\bar{X}_1+\bar{x}_1)2b}\right\}exp\left\{\frac{a(\bar{X}_2-\bar{x}_2)}{a(\bar{X}_2+\bar{x}_2)2b}\right\} \quad (12)$$

The above estimator can be used to create many estimators by changing the values of a, and b. In this case, the generalizing constants a and b can take on any appropriate value or any known parameter of the population, whereas the minimizing constants k, L and M whose values are found by minimizing the MSE. The above Eq. (12) can be expressed in terms of sampling errors as follows in order to get the estimator's MSE.

$$T_s = \left[\left\{k\bar{Y}(1 + e_0) + L\left(\bar{X}_1 - \bar{X}_1(1 + e_1)\right) + M\left(\bar{X}_2 - \bar{X}_2(1 + e_2)\right)\right\}exp\left\{\frac{a(\bar{X}_1-\bar{X}_1(1+e_0))}{a(\bar{X}_1+\bar{X}_1(1+e_0))2b}\right\}exp\left\{\frac{a(\bar{X}_2-\bar{X}_2(1+e_2))}{a(\bar{X}_2+\bar{X}_2(1+e_0))2b}\right\}\right] \quad (13)$$

In terms of sampling errors, the following equation is found by simplifying Eq. (13) and applying the Taylor series while ignoring the high order.

$$T_s = \left[k\bar{y} + k\bar{y}e_0 - \left(L\bar{X}_1 + \frac{k\bar{y}\theta_1}{2}\right)e_1 - \left(M\bar{X}_2 + \frac{k\bar{y}\theta_2}{2}\right)e_2 - \frac{k\bar{y}\theta_1}{2}e_0e_1 - \frac{k\bar{y}\theta_2}{2}e_0e_2 + \left(\frac{M\bar{X}_2\theta_1}{2} + \frac{L\bar{X}_1\theta_2}{2} + \frac{k\bar{y}\theta_1\theta_2}{4}\right)e_1e_2 + \left(\frac{L\bar{X}_1\theta_2}{2} + \frac{3}{8}k\bar{y}_{\theta_1}^2\right)e_1^2 + \left(\frac{M\bar{X}_2\theta_2^2}{2} + \frac{3}{8}k\bar{y}\theta_2^2\right)e_2^2\right] \quad (14)$$

Similarly, the above equation we can write in Eq. (15)

$$[T_s - \bar{Y}] = E\left[k\bar{Y} - \bar{Y} + k\bar{Y}e_0 - \left(L\bar{X}_1 + \frac{k\bar{Y}\theta_1}{2}\right)e_1 - \left(M\bar{X}_2 + \frac{k\bar{Y}\theta_2}{2}\right)e_2 - \frac{k\bar{Y}\theta_1}{2}e_0e_1 - \frac{k\bar{Y}\theta_2}{2}e_0e_2 + \left(\frac{M\bar{X}_2\theta_1}{2} + \frac{L\bar{X}_1\theta_2}{2} + \frac{k\bar{Y}}{4}\theta_1\theta_2\right)e_1e_2 + \left(\frac{L\bar{X}_1\theta_1}{2} + \frac{3}{8}k\bar{Y}\theta_1^2\right)e_1^2 + \left(\frac{M\bar{X}_2\theta_1}{2} + \frac{3}{8}k\bar{Y}\theta_2^2\right)e_2^2\right]$$

(15)

The following bias expression, as in Eq. 16, is obtained by taking the expectation on Eq. (15)

$$Bias(T_s) = k\bar{Y} - \bar{Y} - \frac{k\bar{Y}\theta_1}{2}\lambda C_{yx_1} - \frac{k\bar{Y}\theta_2}{2}\lambda C_{yx_2} + \left(\frac{M\bar{X}_2\theta_1}{2} + \frac{L\bar{X}_1\theta_2}{2} + \frac{k\bar{Y}}{4}\theta_1\theta_2\right)\lambda C_{x_1x_2} + \left(\frac{L\bar{X}_1\theta_1}{2} + \frac{3}{8}k\bar{Y}\theta_1^2\right)\lambda C_{x_1}^2 + \left(\frac{M\bar{X}_2\theta_1}{2} + \frac{3}{8}k\bar{Y}\theta_2^2\right)\lambda C_{x_2}^2 \quad (16)$$

By taking square of Eq. (15) we obtain the below in Eq. (17)

$$(T_s - \bar{Y})^2 = \bar{Y} + k^2\bar{y}^2(1 + e_0^2 + \theta_1^2e_1^2 + \theta_2^2e_2^2 - 2\theta_1e_0e_1 - 2\theta_2e_0e_2 + \theta_1\theta_2e_1e_2) + L^2\bar{X}_1^2e_1^2 + M^2\bar{X}_2^2e_2^2 - k\bar{y}^2\left(2 - \theta_1e_0e_1 - \theta_2e_0e_2 + \frac{1}{2}\theta_1\theta_2e_1e_2 + \frac{3}{4}\theta_1^2e_1^2 + \frac{3}{4}\theta_2^2e_2^2\right) - L\bar{X}_1\bar{y}(\theta_1e_1^2 + \theta_2e_1e_2) - M\bar{y}\bar{X}_2(\theta_2e_2^2 + \theta_1e_1e_2) + kL\bar{X}_1\bar{y}(2\theta_1e_1^2 + 2\theta_2e_1e_2 - 2e_0e_1) + M\bar{y}\bar{X}_2(2\theta_2e_2^2 + 2\theta_1e_1e_2 - 2e_0e_2) + 2LM\bar{X}_1\bar{X}_2e_1e_2 \quad (17)$$

Now by taking expectation of the equation (17) we have the MSE expression in Eq. (18) as

$$MSE(T_s) = \bar{Y}^2 + k^2\bar{Y}^2\left(1 + \lambda C_y^2 + \theta_1^2\lambda C_{x_1}^2 + \theta_2^2\lambda C_{x_2}^2 - 2\theta_1\lambda C_{yx_1} - 2\theta_2\lambda C_{yx_2} + \theta_1\theta_2\lambda C_{x_1x_2}\right) + L^2\bar{X}_1^2\lambda C_{x_1}^2 + M^2\bar{X}_2^2\lambda C_{x_2}^2 - k\bar{Y}^2\left(2 - \theta_1\lambda C_{yx_1} - \theta_2\lambda C_{yx_2} + \frac{1}{2}\theta_1\theta_2\lambda C_{x_1x_2} + \frac{3}{4}\theta_1^2\lambda C_{x_1}^2 + \frac{3}{4}\theta_2^2\lambda C_{x_2}^2\right) - L\bar{X}_1\bar{Y}\left(\theta_1\lambda C_{x_1}^2 + \theta_2\lambda C_{x_1x_2}\right) - M\bar{Y}\bar{X}_2\left(\theta_2\lambda C_{x_2}^2 + \theta_1\lambda C_{x_1x_2}\right) + 2kL\bar{X}_1\bar{Y}\left(2\theta_1\lambda C_{x_1}^2 + 2\theta_2\lambda C_{x_1x_2} - 2\lambda C_{yx_1}\right) + 2kM\bar{Y}\bar{X}_2\left(2\theta_2\lambda C_{x_2}^2 + 2\theta_1\lambda C_{x_1x_2} - 2\lambda C_{yx_2}\right) + 2LM\bar{X}_1\bar{X}_2\lambda C_{x_1x_2} \quad (18)$$

The equation has been simplified, and its simplified form is presented in Equation (19) as follows.

$$\mathrm{MSE}(T_s) = \bar{Y}^2 + k^2\bar{Y}^2A + L^2\bar{X}_1^2B + M^2\bar{X}_2^2C - k\bar{Y}^2D - L\bar{X}_1\bar{Y}E - M\bar{Y}\bar{X}_2F + 2kL\bar{X}_1\bar{Y}G + 2kM\bar{Y}\bar{X}_2H + 2LM\bar{X}_1\bar{X}_2I \quad (19)$$

Also in the above Eq (19) where $A = \left(1 + \lambda C_y^2 + \theta_1^2\lambda C_{x_1}^2 + \theta_2^2\lambda C_{x_2}^2 - 2\theta_1\lambda C_{yx_1} - 2\theta_2\lambda C_{yx_2} + \theta_1\theta_2\lambda C_{x_1x_2}\right)$, $B = \lambda C_{x_1}^2$, $C = \lambda C_{x_2}^2$, $D = \left(2 - \theta_1\lambda C_{yx_1} - \theta_2\lambda C_{yx_2} + \frac{1}{2}\theta_1\theta_2\lambda C_{x_1x_2} + \frac{3}{4}\theta_1^2\lambda C_{x_1}^2 + \frac{3}{4}\theta_2^2\lambda C_{x_2}^2\right)$, $E = \left(\theta_1\lambda C_{x_1}^2 + \theta_2\lambda C_{x_1x_2}\right)$, $F = \left(\theta_2\lambda C_{x_2}^2 + \theta_1\lambda C_{x_1x_2}\right)$, $G = \left(2\theta_1\lambda C_{x_1}^2 + 2\theta_2\lambda C_{x_1x_2} - 2\lambda C_{yx_1}\right)$, $H = \left(2\theta_2\lambda C_{x_2}^2 + 2\theta_1\lambda C_{x_1x_2} - 2\lambda C_{yx_2}\right)$, and $I = \lambda C_{x_1x_2}$

By taking derivative of the MSE epression given in Eq. (18) we obtain the following values for the constants k, L, M.

$k = \frac{BCD-BFH-CEG+EHI+FGI+D}{2ABC+2A-2BH^2-2CG^2+4IGH}$, $L = \frac{(ACE-IAF-CDG+DHI-EH^2+FGH)Y}{2(ABC+A-BH^2-CG^2+2IGH)X}$ and

$M = \frac{Y(ABF-IAE-BDH+DGI+EGH-FG^2)}{2Z(ABC+A-BH^2-CG^2+2IGH)}$

With the above values the minimum MSE of the suggested estimator is given as:

$$MSE(T_s)_{min} = \frac{\varphi_1}{2\varphi_2^2} \quad (20)$$

Where

$$\boldsymbol{\varphi_1} = \Big(Y^2\Big(\Big(-\tfrac{1}{2}CF^2 + 2C^2\Big)G^4 + \Big(((EF-81)C + IF^2)H + CD\Big)IF - CE\Big)\Big)G^3 + \Big(\Big(\Big(4C - \tfrac{F^2}{2}\Big)B - 2IEF - \tfrac{CE^2}{2} - 8\Big)H^2 + 3\Big)\Big(ICE - \tfrac{BCF}{3} + \tfrac{2F}{3}\Big)DH + \Big((CF^2 - 4C^2)B + \tfrac{C^2E^2}{2} + (-IEF-4)C + \tfrac{F^2}{2}\Big)A + \tfrac{CD^2(BC+1)}{2}G^2 + \Big(((EF - 81)B + IE^2)H^3 + 3D\Big(\Big(IF - \tfrac{CE}{3}\Big)B + \tfrac{2E}{3}\Big)\Big)\Big(\Big(((-EF+81)C - IF^2)B - ICE^2 - 3EF + 81\Big)A - I(BC+1)D^2\Big)H - (BC+1)AD(IF-CF)G + (-BH^2 + A(BC+1))\Big(\Big(\tfrac{E^2}{2} - 2B\Big)H^2 - (IE - BF)DH + \Big(\Big(-\tfrac{F^2}{2} + 2C\Big)B + IEF - \tfrac{CE^2}{2} + 2\Big)A - \tfrac{(BC+1)D^2}{2}\Big)$$

and $\boldsymbol{\varphi_2} = (ABC + A - BH^2 - CG^2 + 2IGH)$.

Theoretical comparison

The effectiveness of the proposed estimator can be evaluated by comparing it to the existing estimators using the following conditions.

Condition-1

By comparing Eq (1) and Eq (20), MSE(Ts)<Var(T_1) if

$$[(\lambda C_y^2 - 1) + \delta] > 0 \quad (21)$$

where in the above Eq (21) $\delta = \frac{\varphi_1}{2\varphi_2^2}$

Condition-2

By Comparing Eq (3) with Eq (20), MSE(Ts) < Var(T_2), if

$$\{\lambda\,(C_y^2 + C_{x_1}^2 + C_{x_2}^2 - 2C_{yx_1} - 2C_{yx_2} + 2C_{x_1x_2}) - 1\} + \delta > 0 \quad (22)$$

Condition-3

Comparing Eq (5) with Eq (20), MSE(Ts) <MSE(Tp), if

$$\{\lambda\,(C_y^2 + C_{x_1}^2 + C_{x_2}^2 - 2C_{yx_1} + 2C_{yx_2} - 2C_{x_1x_2}) - 1\} + \delta > 0 \quad (23)$$

Condition-4

by Comparing Eq (7) with Eq (20), MSE(Ts) < MSE(Treg), if

$$\lambda C_y^2(1 - \rho_{yx_1}^2 - \rho_{yx_2}^2 + 2\rho_{yx_1}\rho_{yx_2}\rho_{x_1x_2}) - 1 + \delta > 0 \quad (24)$$

Condition-5

by Comparing Eq (9) with Eq (20), MSE(Ts) < MSE(T_3), if

$$\bar{Y}^2\lambda C_y^2(1 - \rho_{yx}^2) - G_1 - G_2 - 1 + \delta > 0 \quad (25)$$

Condition-6

by Comparing Eq (11) with Eq (20), MSE(Ts) < MSE(T_4),

$$\bar{Y}^2\left[1 - \frac{\varphi_0 + A_6\varphi_1 + A_7\varphi_2}{\varphi}\right] - 1 + \delta > 0 \quad (26)$$

Empirical study

To assess the effectiveness of the proposed estimator on numerical data, three real-world datasets were utilized. Table 2 displays the Mean Squared Errors (MSEs) for both the existing and proposed estimators across all three datasets listed in Table 1. In contrast, Table 3 illustrates the Percentage Relative Efficiencies (PREs) of the estimators compared to the traditional mean estimator. From both tables, it is evident that in all three datasets, the proposed estimators, specifically Ts (1,0) and Ts (1, Cx), outperform the others. Notably, their efficiency is enhanced through the applied transformation. By adjusting the parameters of the auxiliary variables, further improvements in efficiency can be achieved. Therefore, the proposed estimator families prove to be more efficient than the alternatives in estimating the population mean.

Table-1: Data Summary

S. No	Data	N	n	$\bar{Y}^2$	$\bar{X}^2$	$\bar{Z}^2$	C_y	C_x	C_z	ρ_{yx}	ρ_{yz}	ρ_{xz}
1	Source:[29] Wheat-production)Y: in 1974. x: in 1971. z: in 1973.	34	20	856.41	208.88	199.44	0.86	0.72	0.75	0.45	0.45	0.98
2	[Source:[5] y=placebo group x:polio group	34	15	4.92	2.59	2.91	1.012	1.23	1.05	0.73	0.64	0.68

	z:paralytic polio group.											
3	[Source: [1]. y: cultivators x: Area of village, z: households in village.	332	80	1093.1	181.57	143.33	0.763	0.768	0.762	0.97	0.86	0.84

Table-2: Different MSE on Real Data

Estimators	Data-1	Data-2	Data-3
T_1	11168.1	0.9241645	6593.042
T_2	26291.25	2.009312	6755.832
T_p	11858.49	1.282295	7113.921
T_{reg}	11077.64	0.6413499	4764.407
T_3	8794.968	0.3921003	304.5417
T_4	8093.594	0.4150048	547.6293
T_S **(1,0)**	8003.504	0.1091372	18.90024
Ts (1, Cx)	7823.652	0.02793743	18.19442

Table-3: Different PRE's of the Real Datasets on classical estimators

Estimators	Data-1	Data-2	Data-3
T_1	100	100	100
T_2	42.4784	45.99407	97.59039
T_p	94.17817	72.07115	92.67804
T_{reg}	100.8166	144.0968	138.3812
T_3	126.977	235.696	2164.906
T_4	126.9829	222.6876	1203.924
T_S **(1,0)**	139.54	846.7917	34883.37
Ts (1, Cx)	153.33	3307.979	36236.62

Results and Conclusion

In this research, we propose two new families of exponential-type estimators designed specifically for use in simple random sampling, where two auxiliary variables are incorporated. These estimators are intended to improve the accuracy of population parameter estimates, particularly the population mean. To understand how well these estimators, perform. We conducted a detailed analysis using first-order approximation methods. This enabled us to derive important mathematical expressions related to the estimators' key characteristics, such as their bias (which measures systematic error) and Mean Squared Error (MSE), a common measure of an estimator's overall accuracy. During our investigation, we identified specific conditions under which the new estimator families outperform other existing estimators, specifically in terms of lower MSE. Lower MSE values indicate that our proposed estimators provide more accurate and reliable estimates of the population mean compared to competing methods.

To further validate the performance of these estimators, we applied them to real-world datasets. In table-2 and Table-3 the MSE and PRE-are given for all the three datasets. It is obvious from these tables that the MSE values for the proposed T_S (1,0) and Ts (1, Cx) estimators are smaller and PRE-values of the proposed estimators are higher than from all the other contenders to estimates the same parameter. which give us minimum MSE as compared to the existing estimator Through this practical testing, we consistently found that the proposed estimators not only achieved lower MSE values but also delivered higher Percentage Relative Efficiency (PRE) values when compared with traditional estimators for the same population parameters. A higher PRE-indicates that the estimator is more efficient, meaning it requires fewer samples to achieve the same level of accuracy. The combination of lower MSE and higher PRE-consistently demonstrated that the new estimators are more reliable and efficient. This conclusion was supported by both the analysis of actual data and the outcomes of simulation studies, which provided controlled environments to test and compare the estimators.

In summary, the research offers strong evidence that these proposed families of estimators are superior to existing methods. They are more efficient and accurate, making them highly useful in practical applications, particularly when working with simple random sampling and dual auxiliary variables. These findings have significant implications for improving the precision of population estimates in various fields of study.

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