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Generalized Bayesian Estimation for Normal Distribution Based on Fuzzy Information

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Abstract
In nearly all scientific disciplines, the statistical inference about the population relies on the quality of the obtained sampled data. For the statistical inference, the observed sample observations are often recorded as precise numerical values. From the centuries, continuous measurements are recorded as precise numbers, and statistics offer robust techniques and models for translating the sampled raw observations into valuable information. However, the modern metrology sciences recommend that getting precise measurements of continuous phenomena is not very realistic, and measurements inherently entail another form of uncertainty known as fuzziness. Therefore, for optimal inference, it is indispensable to integrate the potential uncertainties in the estimation through recording the continuous measurements by up-to-date fuzzy numbers. The objective of this study is to formulate Bayesian parameter estimators for the Normal distribution, aiming to address both uncertainties i.e. fuzziness and random variation. To completely employ all obtainable uncertainties, the proposed estimators are developed that are based on the informative fuzzy priors and fuzzy observations.

Key Words: Bayesian estimators; Characterizing function; Fuzzy number; Prior information

1 Introduction

Statistics is often labeled as the study of numbers or the science of numerics, helping us understand and utilize real-world data by transforming it into meaningful results. It not only extracts information from real-world data but also allows us to generalize findings from smaller samples to larger populations. This is why statistics is recognized as a fundamental component of the decision sciences. With the progress in computer technology and the widespread availability of statistical software's, it has become essential for conducting research and making knowledgeable decisions across various domains of life. Its role in analyzing data and providing valuable information has made it an essential tool for professionals in diverse fields, contributing significantly to advancements and improvements in decision-making processes (Pardo, 2020).

Probability distribution functions represent the variability in data by illustrating the likelihood of different values. Understanding and predicting patterns within the data is facilitated by these functions.

1.1 Normal Distribution

The normal probability distribution developed by Gauss. C, F., is also known by a Gaussian distribution. The normal probability distribution is symmetric around its mean, meaning that the data is evenly distributed on both sides of the mean. The shape of the normal distribution is often referred to as a bell curve due to its characteristic bell-shaped curve.

Normal distribution is one of the most useful distributions in statistics to model variation in almost every field of life like, economics, biology, engineering, life sciences, physics, and social sciences. Many natural phenomena and measurements tend to follow symmetry that follows a normal distribution.

The central limit theorem is one of the groundbreaking generalizations, which states that for large number of observations the sampling distribution of the average or sum for large number of independent and identically distributed random variables can be approximated by normal distribution, irrespective of the original distribution(s) of the variable(s) (Bryc, 2012).

However, one should note that in the real world not all data perfectly fits a normal distribution, and deviations from normality can have implications for statistical analyses.

Nevertheless, as a result of technological progress, attaining a significant number of observations in the sample to draw conclusions about the population parameter(s) has become increasingly difficult and time-consuming. To improve the precision in estimating parameter(s) based on small samples, the concept of Bayesian probability was introduced in inferential statistics, referred to as Bayesian statistics. The Bayesian statistics integrates the available prior information about the parameter in the estimation. One of the main features of these approaches is the flexibility of updating the prior information if the data changes over time or if the sample size increases. These adaptable properties are playing a vital role in the acceptance of Bayesian statistics across almost every discipline (Heard, 2021).

1.2 Bayesian Statistics

Let T is denoting a continuous random variable that consists of the sample space of a random experiment. The classical statistics is used to model the variation in the said random variables as $T \sim f(\cdot\,|\theta)$, where θ is observed fixed in the parametric space $\Theta \subseteq (-\infty, \infty)$. On the other hand, Bayesian statistics deal(s) the parameter(s) as random variable(s) in the parametric space Θ. Consequently, the parameter will have a known density just like other random variables symbolized by $\Psi(\theta)$ and termed as the *Informative Prior*.

Let $t = (t_1, t_2, \dots, t_n)$ is denoting a precise sample, then the so-called *likelihood function* can be defined as:

$$\mathcal{L}(\theta\,;\underline{t}) = \prod_{i=1}^{n} \mathcal{F}(t_i|\theta) \quad . \qquad (1)$$

Assimilation of the likelihood function and prior density the obtained density is called the *posterior density* of θ, denoted by $\Psi(\theta|t)$, and can be obtained as:

$$\Psi(\theta|\underline{t}) = \frac{\Psi(\theta)\cdot\mathcal{L}(\theta\,;\underline{t})}{\int_\Theta \Psi(\theta)\cdot\mathcal{L}(\theta\,;\underline{t})d\theta} \qquad (2)$$

or in more generalized non-normalized form as:

$$\Psi(\theta|t) \propto \Psi(\theta)\cdot L(\theta\,;t) \qquad (3)$$

for details (Heard, 2021)

2 Generalized Bayesian Estimation and Fuzzy Information

Let assume that $x \sim N(\mu, \sigma 2)\ i.i.d$ having a density:

$$f(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad (4)$$

The parameter μ has prior density of normal defined as $\mu \sim \mathcal{N}(\nu_0\,,\,\sigma_0^2)$,
with density:

$$f(\mu;\nu_0,\sigma_0^2) = \frac{1}{\sqrt{2\pi\sigma_0^2}}\exp\left(-\frac{(\mu-\nu_0)^2}{2\sigma_0^2}\right) \qquad (5)$$

Where σ_0^2 is constant, then using equation (3) the posterior density can be obtained as:

$$f(\mu|\underline{x}) \propto \exp\left(-\frac{(\mu-\nu_0)^2}{2\sigma_0^2}\right)\cdot\exp\left(-\sum\frac{(x-\mu)^2}{2\sigma^2}\right) \qquad (6)$$

which is again a normal density with parameters

$$\mu|x_1,x_2,\dots x_n \sim \mathcal{N}\left(\frac{\sigma_0^2}{\frac{\sigma^2}{n}+\sigma^2}\bar{x} + \frac{\sigma^2}{\frac{\sigma^2}{n}+\sigma^2}\nu_0 \quad , \quad (\frac{1}{\sigma^2}+\frac{n}{\sigma^2})^{-1}\right) \qquad (7)$$

Similarly, if μ is known and $\sigma^2 \sim \mathcal{G}(\alpha\,,\,\beta)$ with density

$$f(\sigma^2;\alpha,\beta) = \frac{\beta^\alpha\sigma^{2\alpha-1}e^{-\beta\sigma^2}}{\Gamma(\alpha)} \qquad \sigma^2 > 0 \qquad (8)$$

then using equation (3), the posterior density is defined as:

$$f(\sigma^2|\underline{x}) \propto \sigma^{2\alpha-1}e^{-\beta\sigma^2}\cdot\exp\left(-\sum\frac{(x-\mu)^2}{2\sigma^2}\right). \qquad (9)$$

which is again a Gamma distribution with the following parameters

$$\sigma^2|x_1,x_2,\dots x_n \sim \mathcal{G}\left(\alpha+\frac{1}{2} \quad , \quad \beta+\frac{1}{2}\sum(x_i-\mu)^2\right) \qquad (10)$$

As advancements in measurement sciences have progressed, it has become evident that achieving precise measurements of continuous phenomena is unattainable, which is a leading reason for the conclusion that the recordings are based on approximated values (Barbato et al., 2013). Furthermore, this leads to the situation, that getting precise measurements of

continuous phenomena is not very realistic, and measurements inherently entail another form of uncertainty known as *fuzziness*.
To overcome this challenge, Zadeh proposed a solution: the generalization of the classical set to what is known as a *fuzzy set* (Zadeh, 1965).

2.1 Fuzzy Number

Let v^* is denoting a *fuzzy number*, determined by *characterizing function* $\psi(\cdot)$, with:
1.0 $\leq \psi(v) \leq 1 \quad \forall v \in \mathbb{R}$
2. The δ-cut $C\delta(v*) := \{v \in \mathbb{R} : \psi(v) \geq \delta\}$ for all $\delta \in (0,1]$ is a finite union of the non-empty compact intervals $[\underline{v}\delta, k, \overline{v}\delta, k]$, i.e.
$C_\delta(v^*) = \bigcup_{k=1}^{K_\delta} [\underline{v}_{\delta,k}, \overline{v}_{\delta,k}] \neq \emptyset$.
3.The bounded support of $\psi(\cdot)$ is defined as, $supp[\psi(\cdot)] := [v \in \mathbb{R} : \psi(v) > 0] \subseteq [\underline{v}\delta, k, \overline{v}\delta, k]$
See (Viertl, 2011).
In accordance with fuzzy set theory, realistic observations exhibit twofold variations: the predominant variation among the observations and fuzziness, a variation of the single observation often ignored. Extensive research has focused on modeling the variability among observations without considering the fuzziness. However, for inference that are more suitable, it is essential to address both types of variations. Upon understanding the prominence of fuzziness in measurements the parameter estimators for some essential distributions are recommended in, (Al-Noor, 2023), (Shah et al., 2022), (Vishwakarma et al., 2018), (Shafiq et al., 2017), (Shapiro, 2013), (Venkatesh and Elango, 2013), (Viertl, 2009), (Wu, 2009), (Lee, 2006), (Nguyen and Wu, 2006), (Hung and Liu, 2004).
In order to make a comprehensive inference, Bayesian statistics combine prior information regarding the unknown parameter with the variability among observations. Subsequently, realizing the importance of fuzziness/ uncertainty in single observations, (Viertl, 1987) proposed to integrate this fuzziness into Bayesian statistics, aiming to comprehend all the available information for optimal inference.

2.2 Bayesian Inference and Fuzzy Information

Let $u_1^*, u_2^*, \ldots, u_n^*$, are denoting fuzzy observations having $F*(u), L*(\alpha, u)$, and $\pi*(\theta)$ fuzzy density function, fuzzy likelihood function, and fuzzy a-priori density function respectively.
Let $\underline{F_\delta}(u)$, $\underline{L_\delta}(\theta, \underline{u}), \underline{\pi_\delta}(\theta)$ are denoting the lower ends of the corresponding family of intervals, while $\overline{F_\delta}(u), \overline{L_\delta}(\theta, \overline{u}), \overline{\pi_\delta}(\theta)$ are denoting the upper ends of the corresponding family of intervals.
Their corresponding lower and upper δ-level curves are denoted as

$$C_\delta\left[\mathcal{F}^*(u)\right] = \left[\underline{\mathcal{F}}_\delta(u)\ ,\ \overline{\mathcal{F}}_\delta(u)\right] \qquad \forall\delta \in (0\ ,\ 1] \tag{11}$$

$$C_\delta\left[\mathcal{L}^*(\theta,\ \underline{u})\right] = \left[\underline{\mathcal{L}}_\delta(\theta\ ,\ \underline{u})\ ,\ \overline{\mathcal{L}}_\delta(\theta\ ,\ \underline{u})\right] \qquad \forall\delta \in (0\ ,\ 1] \tag{12}$$

$$C_\delta\left[\pi^*(\theta)\right] = \left[\underline{\pi}_\delta(\theta)\ ,\ \overline{\pi}_\delta(\theta)\right] \qquad \forall\delta \in (0\ ,\ 1] \tag{13}$$

Based on the fuzzy likelihood and fuzzy prior density the generalized fuzzy Bayes theorem is written as

$$\pi^*(\theta|\underline{u}) = \frac{\pi^*(\theta) \cdot \mathcal{L}^*(\theta\,;\underline{u})}{\int_\Theta \pi^*(\theta) \cdot \mathcal{L}^*(\theta\,;\underline{u})d\alpha} \tag{14}$$

having lower and upper δ-level curves $\underline{\pi_\delta}(\theta|\underline{u})$ and $\overline{\pi_\delta}(\theta|\underline{u})$ respectively.
Where

$$\underline{\pi}_\delta(|\underline{u}) = \frac{\underline{\pi}_\delta(\theta) \cdot \underline{\mathcal{L}}_\delta(\theta\,;\underline{u})}{\int_\Theta \overline{\pi}_\delta(\theta) \cdot \overline{\mathcal{L}}_\delta(\theta\,;\underline{u})d\theta} \qquad \forall\delta \in (0\ ,\ 1] \tag{15}$$

and

$$\overline{\pi}_\delta(\theta|\underline{u}) = \frac{\overline{\pi}_\delta(\theta) \cdot \overline{\mathcal{L}}_\delta(\theta\,;\underline{u})}{\int_\Theta \underline{\pi}_\delta(\theta) \cdot \underline{\mathcal{L}}_\delta(\theta\,;\underline{u})d\theta} \qquad \forall\delta \in (0\ ,\ 1] \tag{16}$$

for detail see (Viertl, 2011)
Using the fuzzy Bayesian approach based on fuzzy information, there have been efforts done to address the issue of fuzziness in available information like,

(Viertl and Hule, 1991), (Fru¨hwirth-Schnatter, 1993), (Wu, 2004), (Viertl and Hareter, 2004), (Huang et al., 2006), (Go¨rkemli and Ulusoy, 2010), (Viertl and Sunanta, 2013), (Pak et al., 2013), (Shafiq, 2017).
Therefore, this effort has been made to generalize the parameters estimators for the most popular distribution used in almost every field of science i.e. Normal distribution.
These approaches will integrate fuzziness of the observations as well as fuzziness of the a-priori density to obtain the more realistic inference.
In this section generalized Bayesian estimators based on informative and non-informative priors are suggested to integrate both the uncertainties present in lifetime observations.
From the literature is clear that lifetime observations are no more precise number but fuzzy. For the priors both the situations are considered, i.e., precise prior information and fuzzy prior information.
From the fuzzy posterior density, the generalized fuzzy parameter estimator can be defined as:

$$C_\delta[\hat{\mu}^*] = \left[\underline{\mu}_\delta \quad , \quad \overline{\mu}_\delta\right] \qquad \forall\delta \in (0\,,\,1] \tag{17}$$

Where

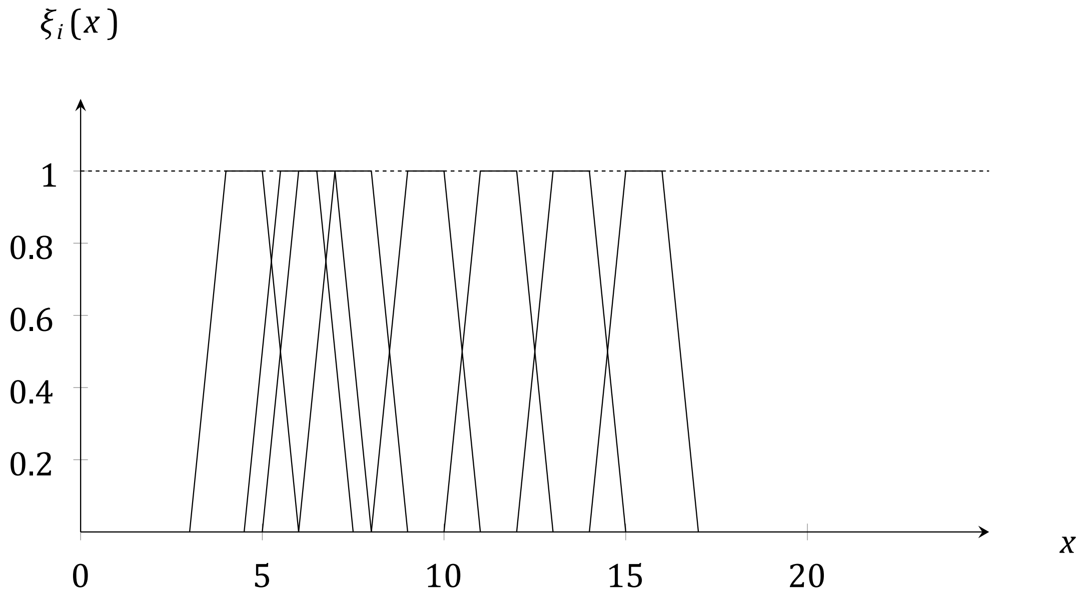
$$\underline{\mu}_\delta = \inf_{x_{1\delta}, x_{2\delta} \ldots x_{n\delta} \in \times_{i=1}^{n} C_\delta(x_{1,i}^*)} \left(\frac{\sigma_{0\delta}^2}{\frac{\sigma^2{}_\delta}{n} + \sigma^2{}_\delta} \bar{x}_\delta + \frac{\sigma^2{}_\delta}{\frac{\sigma^2{}_\delta}{n} + \sigma^2{}_\delta} \nu_{0\delta} \right) \qquad \forall\delta \in (0\,,\,1] \tag{18}$$

and

$$\overline{\mu}_\delta = \sup_{x_{1\delta}, x_{2\delta} \ldots x_{n\delta} \in \times_{i=1}^{n} C_\delta(x_{1,i}^*)} \left(\frac{\sigma_{0\delta}^2}{\frac{\sigma^2{}_\delta}{n} + \sigma^2{}_\delta} \bar{x}_\delta + \frac{\sigma^2{}_\delta}{\frac{\sigma^2{}_\delta}{n} + \sigma^2{}_\delta} \nu_{0\delta} \right) \qquad \forall\delta \in (0\,,\,1] \tag{19}$$

The sample comprises 8 fuzzy observations, each observation is identified by a characterizing function. These characterizing functions enable the clear depiction of these fuzzy observations, as presented in given below Figure 1.

Figure 1: Sample of fuzzy observations



Given below figure 2 is characterizing function of the fuzzy Bayesian estimator of the mean estimator μ^* based on the lower end upper ends of the generating families of intervals obtained from equation (18) and (19).

Figure 2: Characterizing function of the Bayesian estimate of mean μ*

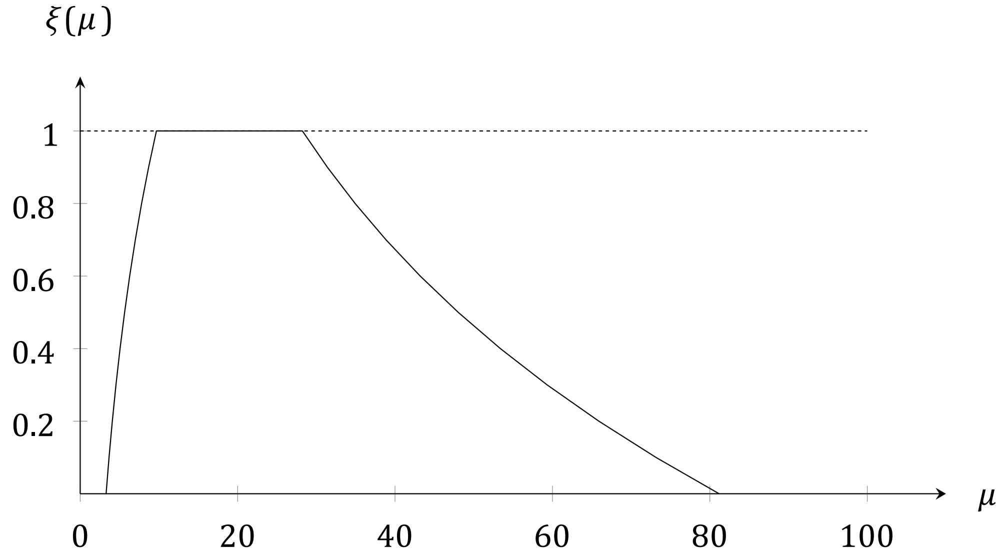


Figure 2 depicts the characterizing function indicates that the parameter μ ranges from 9.68 to 28.1 with a membership degree of 1. Likewise, it ranges from 5.64 to 48.03 with a membership degree of 0.5. Similarly, the lower and upper bounds can be determined for any membership degree within the
interval [0, 1].

The spread of the characterizing function indicates a notably higher degree of fuzziness in the estimator as compared to the data. This implies that even minor degrees of fuzziness lead to increased uncertainty in the estimates. Hence, it authenticates a convincing rationale for considering the fuzziness of the observations for statistical inference.

For the Bayesian estimator of variance of the normal distribution for the fuzzy observation, using equation (10) we obtain as:

$$\sigma^{2*}|x_1^*, x_2^*, ...x_n^* \sim \mathcal{G}\left(\alpha^* + \frac{1}{2} \quad , \quad \beta^* + \frac{1}{2}\sum(x_i^* - \mu^*)^2\right) \tag{20}$$

where

$$\underline{\sigma^2}_\delta = \inf_{x_{1\delta}, x_{2\delta} \ldots x_{n\delta} \in \times_{i=1}^n C_\delta(x_{1,i}^*)} \left(\mathcal{G}\left(\alpha^* + \frac{1}{2} \quad , \quad \beta^* + \frac{1}{2}\sum(x_i^* - \mu^*)^2\right)\right) \qquad \forall\delta \in (0\,,\,1] \tag{21}$$

and

$$\overline{\sigma^2}_\delta = \sup_{x_{1\delta}, x_{2\delta} \ldots x_{n\delta} \in \times_{i=1}^n C_\delta(x_{1,i}^*)} \left(\mathcal{G}\left(\alpha^* + \frac{1}{2} \quad , \quad \beta^* + \frac{1}{2}\sum(x_i^* - \mu^*)^2\right)\right) \qquad \forall\delta \in (0\,,\,1] \tag{22}$$

defining α and β MLEs of the gamma distribution, these were further simplified in the following equations:

$$\underline{\hat{\alpha}}_\delta \,,\, \underline{\hat{\beta}}_\delta = \min_{x_{1\delta}, x_{2\delta} \ldots x_{n\delta} \in \times_{i=1}^n C_\delta(x_{1,i}^*)} \left(\mathcal{G}\left(\alpha \quad , \quad \beta\right)\right) \qquad \forall\delta \in (0\,,\,1] \tag{23}$$

And

$$\overline{\hat{\alpha}}_\delta \,,\, \overline{\hat{\beta}}_\delta = \max_{x_{1\delta}, x_{2\delta} \ldots x_{n\delta} \in \times_{i=1}^n C_\delta(x_{1,i}^*)} \left(\mathcal{G}\left(\alpha \quad , \quad \beta\right)\right) \qquad \forall\delta \in (0\,,\,1] \tag{24}$$

Using the above equations lower $\underline{\sigma}_\delta^2$ and upper $\overline{\sigma}_\delta^2$ ends of the generating family of intervals of Bayesian fuzzy estimator $\sigma^{2*}$are obtained as:

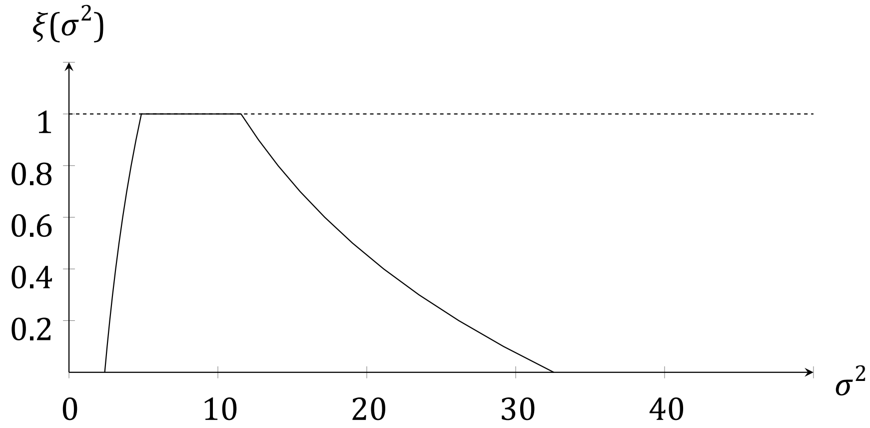
$$\underline{\sigma^2}_\delta = \frac{\underline{\hat{\alpha}}_\delta + 1/2}{\overline{\hat{\beta}}_\delta + n/2\max(\sigma^2)} \qquad \forall\delta \in (0\,,\,1] \tag{25}$$

And

$$\overline{\sigma^2}_\delta = \frac{\overline{\hat{\alpha}}_\delta + 1/2}{\underline{\hat{\beta}}_\delta + n/2\min(\sigma^2)} \qquad \forall\delta \in (0\,,\,1] \tag{26}$$

Using the generating family of intervals $\mathcal{F}_\delta$ defined as $\left(\mathcal{F}_\delta = (\underline{\sigma^2}_\delta\,,\, \overline{\sigma^2}_\delta) \quad \forall \delta \in (0\,,\,1]\right)$, the characterizing function of the suggested fuzzy Bayesian estimator is predicted in figure 3 below:

Figure 3: Characterizing function of the Bayesian estimate of variance σ^{2*}



Similarly, Figure 3 illustrates the characterizing function, showing that the parameter σ^{2*} spans from 4.86 to 11.55 with a full membership degree of 1, and from 3.36 to 19.06 with a membership degree of 0.5. Additionally, the lower and upper bounds can be established for any membership degree within the range of [0, 1].
The extent of fuzziness displayed by the characterizing function signifies a significantly greater level of uncertainty in the estimator compared to the data. This suggests that even slight degrees of fuzziness result in heightened uncertainty in the estimates. Consequently, it provides a compelling justification for taking into account the fuzziness of observations in statistical inference

3 Conclusion

The prime objective of the data analysis is to obtain more realistic inferences about the population parameter(s) from the sample observations. Statistics provides sophisticated techniques for making statistical decisions about the population based on sample data. In most situations, the variable(s) under study are of a continuous nature, but the observations are measured in the form of precise numbers. But according to advanced measurement science, continuous variables can't be measured precisely, and by doing so, we may draw misleading inference. Therefore, for the best possible inference, it is noteworthy to handle all possible uncertainties. Hence, the continuous variables should be measured and recorded by up-to-date fuzzy numbers.
Since the Normal distribution is among one of the most used distributions in real life applications in almost every field of life. In the same way, precise measurements are not possible for any kind of continuous phenomenon. Therefore, an effort is made to generalize the Bayesian estimators for the parameter of the Normal distribution to cover both uncertainties. The Bayesian inference based on the suggested estimators are much better than the classical one, because these estimators utilize all the available information in the form of variation and fuzziness. In addition to these, the obtained estimators have more fuzziness as compared to the data. Therefore, it strongly supports the claim to include fuzziness of the single observations in statistical inference.

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