Kurdish Studies
September 2024
Volume: 12, No: 5, pp1030-1040
ISSN: 2051-4883 (Print) | ISSN 2051-4891 (Online)
www.KurdishStudies.net

DOI: 10.53555/ks.v12i5.3397

A New Flexible Exponential Type Family Of Distributions: Application With COVID-19 Data And Simulation

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Abstract
Since December 2019, every aspect of daily life has been profoundly impacted by the COVID-19 pandemic. The coronavirus spread over 200 countries worldwide and killed more than 5 million individuals. Modeling the death rate due to COVID-19 has become vital in guiding health authorities in implementing effective policies and decision-making. This research investigates the New Flexible Exponential Type Family (FETF) of distributions and explores a specific distribution within this family, the Exponential Type Weibull distribution (ETW) that can be used to model the mortality rate COVID-19 patients. The research thoroughly explores ETW characteristics, such as order statistics, moments, survival function, hazard function, mean residual life function, and quantile function. The widely accepted likelihood approach is employed to estimate the unknown parameters in the suggested model. The recommended model is carefully tested against real-world non-monotonic COVID-19 data and simulated datasets to determine its usefulness. The results exhibit the superior performance of the recommended model when compared to several prominent alternatives.

Keywords: Moment generating function, Quantile function, Mode, Order statistics, Shannon entropy, Weibull distribution, COVID-19.

1. Introduction

The Weibull (Wb) distribution is a very well-liked lifetime distribution due to its pliability in fitting failure times. Lifetime data are found in different domains such as survival analysis, reliability engineering, medicine, economics, business, and many others. However, the Wb distribution has limitations in that it models only monotonic hazard rates. Hence, it's unsuitable to model the non-monotonic hazard rate like bathtub shape. To overcome this constraint, numerous researchers have adapted the Weibull distribution to account for hazard rates that are not strictly monotonic. Many scholars have awarded significant revivals of these modifications, for instance, Sarhan and Zaindin (2009) studied the "Modified Weibull distribution" and discussed their properties. Pal and Tienuwan (2014) introduced a beta-transmuted Weibull distribution and studied their numerous mathematical properties. Abid and Abdulrazak (2017) worked on a truncated Frechet-G family of distribution. Ahamad & Ghazal (2020) discussed the exponential additive Weibull distribution with five parameters. Oluyede et al. (2021) introduced a novel family termed the Exponentiated Half Logistic-Power Generalized-G distribution and analyzed four cases for the proposed distribution. Recently, Ghazal (2023) introduced an innovative distribution referring to NMW3 distribution, which is a creative addition to the three-parameter-modified Weibull distribution. Especially, the NMW3 distribution exhibits hazard rate shapes with either an ascending, descending or bathtub-shaped structure. The author carried out a detailed empirical examination, involving a comparison with existing lifetime distributions, to evaluate the effectiveness and applicability of this recently proposed model. Jiang et al. (2023) carried out a new version of the Wb distribution known as the "Improved Modified Weibull Distribution" (IMW). This distribution exhibits noticeable flexibility in its capacity to efficiently model the three distinctive stages characterizing the modified bathtub-shaped hazard function. Shama et al. (2023) investigated a modified duplicate of the Wb distribution, known as the "Modified Generalised Weibull Distribution." This update incorporates four parameters and demonstrates the ability to appropriately express a hazard rate function represented by a bathtub pattern. Khan et al. (2023) proposed a new statistical model called the "New Beta Power Very Flexible Weibull Distribution." This model stands out for its capacity to properly accommodate and describe five distinct failure rate patterns. Alomair et al. (2023) enhanced the probability models using the trigonometry technique. The new variant of the Wb distribution is known as a type-I cosine exponentiated Weibull (TICE-Weibull) distribution. By using two actual data sets, the model's flexibility is assessed. Tashkandy and Emam (2023) introduced the Exponential Weibull distribution (E-WD), which represents a novel extension of the three-parameter Weibull model. This proposed probability model provides a flexible structure that may accommodate a variety of data distributions, including positively skewed, negatively skewed, decreasing, curving, and symmetric shapes. Alotaibi et al. (2023) suggest a modified Weibull distribution by integrating an additional shape parameter via the modified alpha power transformation technique. Frequently, this model is denoted as the "Weibull distribution with a modified alpha-power transformation". This unique distribution's beauty and significance arise from its capability to capture both monotonic and non-monotonic failure rate curves effectively. El-Monsef et al. (2022) investigated a novel statistical model termed Poisson Modified Weibull Distribution (PMW). This probability model was initiated to achieve a hazard rate function with a variety of patterns, particularly the bathtub shape. In the field of reliability analysis, this unique form is quite useful. Aljarrah et al. (2024) worked on the generalization of lifetime distribution and introduced a novel class of distributions termed as the generalized symmetric T-Y {Y} class. To construct this class of distribution, authors used the quantile function of the generalized Weibull distribution. They also introduced a sub-family of distributions. First is the generalized symmetric t-distribution and the second is the generalized Symmetric Exponential-R family of distributions. Abdelall et al. (2024) proposed a new generator called the "extended odd inverse Weibull-generator", which can be used to generate a new continuous probability distribution. Using this generator, a special case ware introduced by incorporating the Weibull model as a baseline distribution is called extended odd inverse Weibull-G family which is extended to five parameters. To analyze the behavior of parameters, a Bayesian estimation method was used and compared with the ordinary MLE method. Results show that the Bayesian approach performs well against MLE. Rahman (2024) introduced a new family of distribution called Triangle-G (TR-G) family of distribution. Also, some properties of the TR-G family have been derived from this research. Using the Inverse Weibull distribution of parameters two and three, the special members of the TR-G family of distributions are derived called TR-Inverse Weibull (TR-IW) distributions. Both distributions of parameters two and three show flexibility against some well-known distributions. AbaOud & Almuqrin (2024) modified the three parameters generalized inverse Weibull (GI-Weibull) distribution and developed a new weighted version of the GI-Weibull distribution may called weighted generalized inverse Weibull (WGI-Weibull) distribution. The WGI-Weibull model is considered as an alternative to the GI-Weibull which is best fitted compared to other existing distributions as it's established with medical data. Alzahrani (2024) discussed and modified the Weibull distribution by incorporating extra parameters, resulting in the development of a new distribution called the "Exponential-Weibull Weibull (EWW) distribution." This five-parameter model has the capability to deal with non-monotonic failure rate data. The parameters of the EWW model were estimated both using Bayesian and maximum likelihood procedures. The EWW model is compared with some well-known probability models using the dropout times of the students, which indicates that the proposed model is outperforming. Widyaningsih & Ivana (2024) present a continuous lifetime probability distribution named the Weibull-Poisson (WP) distribution. This hybrid model is formed by combining the Weibull and zero-truncated Poisson distributions. The hazard function, such as monotone decreasing, monotone increasing, or upside-down bathtub, can impact the flexibility of the WP distribution. To demonstrate, the Weibull-Poisson distribution is used for guinea pig survival data following infection with the Turblece virus Bacilli. Lakibul (2023) introduced the idz distribution, which is a mixture of three lifetime distributions, i.e., the Weibull, Exponential and Ailamujia distributions. The idz distribution failure rate function produces three types of failure rates i.e., non-monotonic constant, non-monotonic declining and right-tailed unimodal. The suggested distribution was applied to breast cancer data and compared to some current lifetime distributions. Results show that the Idz distribution provides superior estimates for the dataset. Eghwerido & Agu (2023) developed a novel family of continuous distributions called the Alpha Power Muth or Teissier-G (APTG) family of distributions. The authors also got the reliability measures and the combined progressive type II censoring method. To evaluate the numerical application of the proposed model, they used both simulated and real data. Sundaram & Jayakodi (2023) proposed an alternative distribution to Weibull and Exponential based on q-Exponential-G family of distributions called the q-Exponential-Weibull distribution.

In this study, the Wb distribution is modified, resulting in the emergence of a new class of probability distributions known as the FETF of Distributions (FETF). The special case of the FETF called Exponential Type Weibull distribution (ETW) is also discussed.

2. FETF of Distributions

We present a novel class of probability distributions in this section called Flexible Exponential Type Family (FETF). Take into consideration X, a continuous random variable. This newly defined FETF family's cumulative distribution function (or CDF) is written as follows:

$$F_{FETF}(x) = \begin{cases} e^{\mu-\mu/F(x)}; if \ \mu > 1 \\ e^{1-1/F(x)}; if \ \mu = 1 \end{cases} \quad (1)$$

In this context, μrepresents the scale parameter, and $F(x)$ denotes CDF of the baseline model.

3. ETW Distribution

In this section, we have discussed the special case of FETF. We made some modifications to the traditional Wb distribution CDF and attained the new CDF of the ETW distribution in the form of (3). The Wb distribution CDF is formulated in the following manner.

$$\left(F_w(x)\right)_{x>0} = 1 - e^{-\mu x^{\pi}} \quad (2)$$

Where, $\mu$and π represent the scale and shape parameter, respectively.

$$F_{ETW}(x;\ \mu,\pi) = e^{\mu-\frac{\mu}{1-e^{-\mu x^{\pi}}}};\ such\ that\ x,\mu,\pi > 0 \quad (3)$$

With relation to equation (3), the probability density function (PDF) can be displayed as follows:

$$f_{ETW}(x;\ \mu,\pi) = \frac{\mu^2 \pi x^{\pi-1} e^{\mu x^{\pi}-\frac{\mu}{e^{\mu x^{\pi}}-1}}}{(e^{\mu x^{\pi}}-1)^2}, \text{for } x,\mu,\pi > 0 \quad (4)$$

Figure 1 illustrates the ETW PDF and the CDF graphically for a range of parameter values.

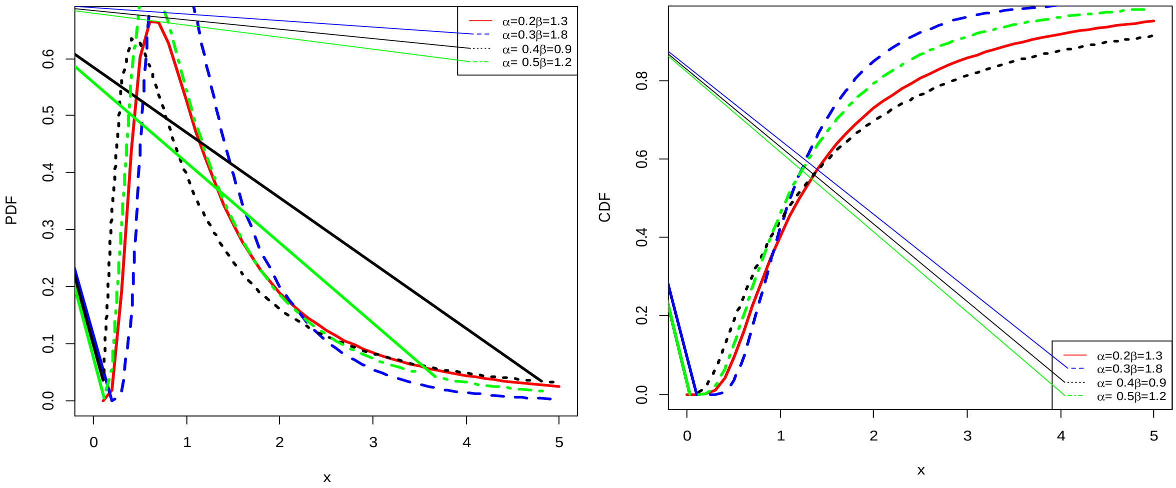

Fig-1: PDF and CDF of the ETW

4. Properties of the ETW Distribution

This section examined a number of statistical aspects of the ETW. These properties covered the survival function, hazard function, r[th] moments, quantile function, and Shannon entropy.

4.1 The Survival and Hazard Rate Function

The survival function of the suggested distribution is defined as follows: if X is a random variable that follows ETW.
$R(x) = 1 - F(x)$, putting equation (3), we have

$$R(x;\ \mu,\gamma) = 1 - e^{\frac{\mu}{e^{\mu x^{\gamma}}-1}} \quad (5)$$

Using equations (4) and (5), we got hazard function H(x) in the form of (6) as H(x) is the ratio of PDF to Survival function.

$$\mathrm{H}(x;\mu,\pi) = \frac{\mu^2 \pi x^{\pi-1} e^{\mu x^{\pi}}}{\left(e^{\frac{\mu}{e^{\mu x^{\pi}}-1}}-1\right)\left(e^{\mu x^{\pi}}-1\right)^2} \quad (6)$$

Figure 2 shows how the hazard rate function varies for various parameter values.

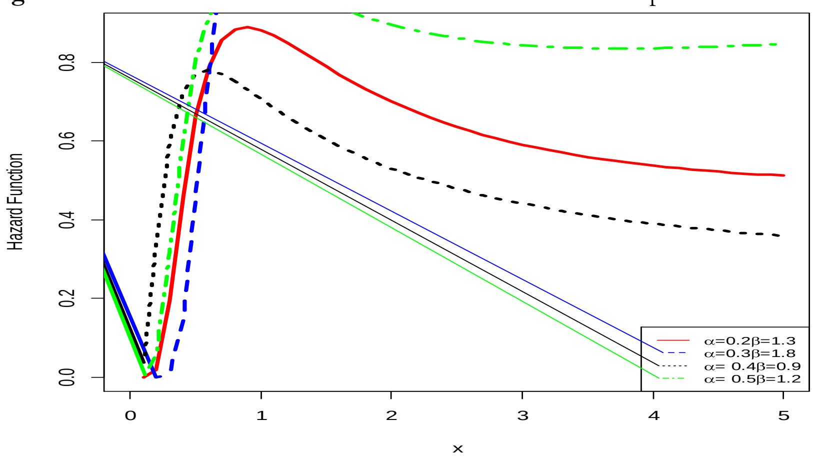

Fig-2: Hazard Function of ETW

4.2 Mean Residual Life (MRL) Function

The MRL of the ETW distribution with parameter $\mu, \gamma$is defined as

$$M_R(t) = \frac{\int_t^{\infty} x f(x) dx}{R(x) - t} \quad (7)$$

Substituting eq(4) and eq(5) in the above expression, we have

$$M_R(t) = \frac{\int_t^{\infty} x \frac{\mu^2 \pi x^{\pi-1} e^{\mu x^{\pi} - \frac{\mu}{e^{\mu x^{\pi}}-1}}}{\left(e^{\mu x^{\pi}}-1\right)^2} dx}{1 - e^{\frac{\mu}{e^{\mu x^{\pi}}-1}} - t} \quad (8)$$

by solving the numerator of equation (8), we have

$$-\left(\frac{1}{\mu}\right)^{\frac{1}{\pi}} \left(-\sum_{k=1}^{\infty} \frac{(-1)^k}{k}\right)^{\frac{1}{\pi}} \mu^{\frac{k}{\pi}} \frac{}{)\left(-\frac{k-\pi}{\pi}, z\right)}$$

It was assuming that $\pi(k - 2\pi) \neq 0, \pi(k - \pi) \neq 0$

After simplification and substituting the above expression in (8), we obtained the MRL in the form

$$M_R(t) = \frac{-\left(\frac{1}{\mu}\right)^{\frac{1}{\pi}}\left(-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\right)^{\frac{1}{\pi}}\mu^{\frac{k}{\pi}}\frac{}{)\left(-\frac{k-\pi}{\pi},z\right)}}{1-t-e^z}$$

4.3 The rth Moments

Consider a random variable X following the ETW, the r[th] moments about the origin are illustrated as follows.

$$\mu_r = \int_0^\infty x^r f(x)\,dx$$

Putting equation (4), we get

$$\mu_r = \int_0^\infty x^r \frac{\mu^2\pi x^{\pi-1}e^{\mu x^\pi - \frac{\mu}{e^{\mu x^\pi}-1}}}{(e^{\mu x^\pi}-1)^2}dx \qquad (9)$$

Let $z = \frac{\mu}{e^{\mu x^\pi}-1}$ and $-dz = \frac{\mu^2\pi x^{\pi-1}e^{\mu x^\pi}}{(e^{\mu x^\pi}-1)^2}$, substituting these two expressions in (9) which take the form

$$= \int_0^\infty -x^r e^{-z}dz \qquad (10)$$

Where $x = \left[\frac{\log\left(\frac{\mu}{z}+1\right)}{\mu}\right]^{\frac{1}{\pi}}$, then equation (10) becomes.

$$= \left(\frac{1}{\mu}\right)^{\frac{1}{\pi}}\int_0^\infty \left[\log\left(\frac{\mu}{2}+1\right)\right]^{\frac{1}{\pi}} e^{-z}dz$$

Finally, we obtained the result as

$$\mu_r = -\left(-\sum_{k=1}^{\infty}\frac{(-1)^k}{\mu k}\right)^{\frac{1}{\pi}}\mu^{\frac{k}{\pi}}\frac{}{)\left(-\frac{k-\pi}{\pi},z\right)}$$

It was assuming that $\pi(k-2\pi) \neq 0, \pi(k-\pi) \neq 0$

4.4 Order Statistics

Let $X_1, X_2, X_3, \ldots. X_n$ be ordered random variables from ETW, then the PDF of the i_{th} order statistic is given as

$$f_{x(i)}(x) = \frac{n!}{(i-1)!(n-i)!}F(x)^{i-1}[1-F(x)]^{n-i}f(x) \qquad (11)$$

Using equation (3) and (4), equation (11) takes the form.

$$f_{x(i)}(x) = \frac{n!}{(i-1)!\,(n-i)!}\left(e^{-\frac{a}{e^{\mu x^\pi}-1}}\right)^{i-1}\left(1-e^{-\frac{\mu}{e^{\mu x^\pi}-1}}\right)^{n-i}\frac{\mu^2\pi x^{\pi-1}e^{\mu x^\pi-\frac{\mu}{e^{\mu x^\pi}-1}}}{(e^{\mu x^\pi}-1)^2}$$

The smallest order statistic of the ETW distribution is given as

$$f_{x(1)}(x) = \frac{n\mu^2\pi x^{\pi-1}e^{\mu x^\pi-\frac{\mu}{e^{\mu x^\pi}-1}}}{(e^{\mu x^\pi}-1)^2}\left(1-e^{-\frac{\mu}{e^{\mu x^\pi}-1}}\right)^{n-1}$$

And the largest order statistic of the ETW distribution is given as

$$f_{x(n)}(x) = \frac{n\mu^2\pi x^{\pi-1}e^{\mu x^\pi-\frac{\mu}{e^{\mu x^\gamma}-1}}}{(e^{\mu x^\pi}-1)^2}\left(e^{-\frac{\mu}{e^{\mu x^\pi}-1}}\right)^{n-1}$$

4.5 Quantile Function and Median

With $u(0,1)$ representing a uniform random number and $F_{ETW}(x)$ representing the CDF of the suggested distribution in (3), which can be solved for the random variable X, the quantile function is represented as $F_{ETW}(x) = u$. It is defined as the quantile function.

$$e^{\mu\left(1-\frac{1}{1-e^{-\mu x^\pi}}\right)} = u \qquad (12)$$

Taking log on both side of the above expression, we get

$$\mu\left(1-\frac{1}{1-e^{-\mu x^\pi}}\right) = \log(u)$$

After solving the above expression for X, we get the quantile function as.

$$x = \left[\frac{\log\left(1-\frac{\mu}{\log(u)}\right)}{\mu}\right]^{\frac{1}{\pi}} \qquad (13)$$

To get median put $u = \frac{1}{2}$ in equation (13), we have.

$$Median\ (x) = \left[\frac{\log\left(1+\frac{\mu}{\log(2)}\right)}{\mu}\right]^{\frac{1}{\pi}}$$

4.6 The Skewness and Kurtosis

The Quantiles approach is utilized to evaluate the distribution's shape, including skewness and kurtosis. Faton and Ibrahim derived the Bowley's skewness formula [9] and Kurtosis formula by Moors [10] are given as

$$Sk = \frac{Qun\left(\frac{3}{4}\right) + Qun\left(\frac{1}{4}\right) - 2Qun\left(\frac{1}{2}\right)}{Qun\left(\frac{3}{4}\right) - Qun\left(\frac{1}{4}\right)}$$

$$Kr = \frac{Qun\left(\frac{7}{8}\right) + Qun\left(\frac{3}{8}\right) - Qun\left(\frac{5}{8}\right) - Qun\left(\frac{1}{3}\right)}{Qun\left(\frac{6}{8}\right) - Qun\left(\frac{2}{8}\right)}$$

Table 1 presents the skewness and kurtosis values of the ETW distribution across various parameter settings.

Table-1: Skewness and Kurtosis

Parameters	Skewness	Kurtosis
$\mu = 0.2, \pi = 0.5$	0.645918	2.834901
$\mu = 0.3, \pi = 0.7$	0.483295	1.905418
$\mu = 1.2, \pi = 1.8$	0.148872	1.248325
$\mu = 2.2, \pi = 3.1$	0.071233	1.217026
$\mu = 3.8, \pi = 4.2$	0.044875	1.219259
$\mu = 4.9, \pi = 5.3$	0.034914	1.221272

4.7 Mode

We can get the mode of ETW by taking derivative of equation (4) and equate them zero, then solve for X

$$f(x)^{/} = 0$$

$$\frac{d}{dx}\left(\frac{\mu^2\pi x^{\pi-1}e^{\mu x^{\pi}-\frac{\mu}{e^{\mu x^{\pi}}-1}}}{(e^{\mu x^{\pi}}-1)^2}\right) = 0$$

$$-\frac{\mu^{\pi x^{\pi-2}}\left[(\mu\pi x^{\pi}-\pi+1)e^{2\mu x^{\pi}} + (-\mu^2\pi x^{\pi}+2\pi-2)e^{\mu x^{\pi}} - \mu\pi x^{\pi}-\pi+1\right]e^{\mu x^{\pi}-\frac{\mu}{e^{\mu x^{\pi}}-1}}}{(e^{\mu x^{\pi}}-1)^4} = 0$$

$$(\mu\pi x^{\pi}-\pi+1)e^{2\mu x^{\pi}} + (-\mu^2\pi x^{\pi}+2\pi-2)e^{\mu x^{\pi}} - \mu\pi x^{\pi}-\pi+1 = 0 \qquad (14)$$

In general, there is not an explicit solution for (14). However, by using iterative procedure one can get a numerical solution.

5. Maximum Likelihood Estimation (MLE)

The conventional method of estimating, known as maximum likelihood estimates, was used to produce the estimates for the unknown parameters in the probability model. Let the independent sample of size n represented ETW distribution are selected. In this way, the likelihood function L is expressed as.

$$L = \prod_{i=1}^{n} f(x_i; \mu, \pi)$$

Putting (4) into previous expression gives us

$$L = \prod_{i=1}^{n} \frac{\mu^2\pi x^{\pi-1}e^{\mu x^{\pi}-\frac{\mu}{e^{\mu x^{\pi}}-1}}}{(e^{\mu x^{\pi}}-1)^2}$$

Log of the above expression yield the log likelihood function as

$$\ell = 2n\log\mu + n\log\pi + (\pi-1)\log\sum_{i=1}^{n} x_i + \mu\sum_{i=1}^{n} x_i^{\pi} - \frac{\mu}{e^{\mu\sum_{i=1}^{n} x_i^{\pi}}-1} - 2\log\sum_{i=1}^{n}\left(e^{\mu x_i^{\pi}}-1\right)$$

The unknown parameters can be determined by taking partial derivatives of ℓ with respect to the parameters μ and, π and setting the results to zero.

$$\frac{\partial\ell}{\partial\mu} = \frac{2n}{\mu} + \frac{\mu\sum_{i=1}^{n} x_i^{\pi} e^{\mu\sum_{i=1}^{n} x_i^{\pi}}}{\left(e^{\mu\sum_{i=1}^{n} x_i^{\pi}}-1\right)^2} - \frac{1}{e^{\mu\sum_{i=1}^{n} x_i^{\pi}}-1} + \sum_{i=1}^{n} x_i^{\pi} + \sum_{i=1}^{n}\frac{2x_i^{\pi}e^{\mu x_i^{\pi}}}{e^{\mu x_i^{\pi}}-1} = 0 \qquad (15)$$

$$\frac{\partial \ell}{\partial \pi} = \frac{n}{\pi} + \log \sum_{i=1}^{n} x_i + \frac{\mu^2 \sum_{i=1}^{n} x_i^{\pi} \log \sum_{i=1}^{n} x_i e^{\mu \sum_{i=1}^{n} x_i^{\pi}}}{\left(e^{\mu \sum_{i=1}^{n} x_i^{\pi}} - 1\right)^2} + \mu \sum_{i=1}^{n} x_i^{\pi} \log \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} \frac{2\mu x_i^{\pi} \log x_i e^{\mu x_i^{\pi}}}{e^{\mu x_i^{\pi}} - 1} = 0 \quad (16)$$

See, equation (15) and (16) are not in closed form. Hence, the exact parameters' values are difficult to estimate. Alternatively, to obtain the MLEs, numerical techniques like the Newton-Raphson and Bisection methods can also be used.

5.1 Asymptotic Confidence Bounds

As previously stated, the unknown parameters do not possess closed-form solutions, making it impossible to derive the correct distribution for the MLEs. However, utilizing asymptotic distribution of MLEs, we have established asymptotic confidence bounds for these unknown values.

The second partial derivates of the equation (15) and (16) are respectively given as

$$I_{11} = \frac{\partial \ell}{\partial \mu^2} = -\left[\frac{\sum_{i=1}^{n} x_i^{\pi} e^{\mu \sum_{i=1}^{n} x_i^{\pi}} \left\{(2n \sum_{i=1}^{n} x_i^{\pi} - 1) e^{\mu \sum_{i=1}^{n} x_i^{\pi}} + 2n \sum_{i=1}^{n} x_i^{\pi} + 1\right\}}{\left(e^{\mu \sum_{i=1}^{n} x_i^{\pi}} - 1\right)^3}\right] + \sum_{i=1}^{n} \frac{2 x_i^{2\pi} e^{\mu x_i^{\gamma}}}{\left(e^{\mu \sum_{i=1}^{n} x_i^{\pi}} - 1\right)^2}$$

$$I_{22} = \frac{\partial \ell}{\partial \pi^2}$$

$$= \frac{\mu \sum_{i=1}^{n} x_i^{\pi} \log \sum_{i=1}^{n} x_i^2 \left[e^{\mu \sum_{i=1}^{n} x_i^{\pi}} \left(e^{2\mu \sum_{i=1}^{n} x_i^{\pi}} - (\mu(\mu \sum_{i=1}^{n} x_i^{\pi} - 1) + 3) e^{\mu \sum_{i=1}^{n} x_i^{\pi}} - \mu(\mu \sum_{i=1}^{n} x_i^{\pi} + 1) + 3\right) - 1\right]}{\left(e^{\mu \sum_{i=1}^{n} x_i^{\pi}} - 1\right)^3}$$

$$- \sum_{i=1}^{n} \frac{2\mu x_i^{\pi} \log x_i^2 e^{\mu x_i^{\pi}} \left(e^{\mu x_i^{\pi}} - \mu x_i^{\pi} - 1\right)}{\left(e^{\mu x_i^{\pi}} - 1\right)^2}$$

$$I_{12} = \frac{\partial \ell}{\partial \mu \pi}$$

$$= \frac{\sum_{i=1}^{n} x_i^{\pi} \log \sum_{i=1}^{n} x_i \left[e^{\mu \sum_{i=1}^{n} x_i^{\pi}} \left(e^{2\mu \sum_{i=1}^{n} x_i^{\pi}} - (\mu(\mu \sum_{i=1}^{n} x_i^{\pi} - 2) + 3) e^{\mu \sum_{i=1}^{n} x_i^{\pi}} - \mu(\mu \sum_{i=1}^{n} x_i^{\pi} + 2) + 3\right) - 1\right]}{\left(e^{\mu \sum_{i=1}^{n} x_i^{\pi}} - 1\right)^3}$$

$$- \sum_{i=1}^{n} \frac{2 x_i^{\pi} \log x_i e^{\mu x_i^{\pi}} \left(e^{\mu x_i^{\pi}} - \mu x_i^{\pi} - 1\right)}{\left(e^{\mu x_i^{\pi}} - 1\right)^2}$$

The observed information matrix can be defined as

$$I = -\begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}$$

And the variance-covariance matrix is given as

$$V = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} \\ I_{21} & I_{22} \end{pmatrix}^{-1}$$

We used the corresponding MLE to replace the parameters to produce the estimate, which may be illustrated as

$$\hat{V} = \begin{pmatrix} \hat{I}_{11} & \hat{I}_{12} \\ \hat{I}_{21} & \hat{I}_{22} \end{pmatrix}^{-1}$$

We can construct the $(1-\beta)100\%$ confidence intervals using the above variance covariance matrix for the parameters μ and π in the form

$$\hat{\mu} \pm Z_{\beta/2} \sqrt{\operatorname{var}(\hat{\mu})}, \quad \hat{\pi} \pm Z_{\beta/2} \sqrt{\operatorname{var}(\hat{\pi})}$$

Where, Z_{β} indicates the upper β^{th} percentile of standard normal distribution.

6. Shannon Entropy

Entropy quantifies the degree of diversity within a given system. One of the famous entropies is Shannon entropy which may be illustrated as

$$S_E(x) = -\int_0^{\infty} f(x) \log f(x) dx \quad (17)$$

Substituting equation (4) in (17), we get

$$S_E(x) = -\int_0^{\infty} \frac{\mu^2 \pi x^{\pi-1} e^{\mu x^{\pi} - \frac{\mu}{e^{\mu x^{\pi}} - 1}}}{\left(e^{\mu x^{\pi}} - 1\right)^2} \log\left(\frac{\mu^2 \pi x^{\pi-1} e^{\mu x^{\pi} - \frac{\mu}{e^{\mu x^{\pi}} - 1}}}{\left(e^{\mu x^{\pi}} - 1\right)^2}\right) dx \quad (18)$$

By taking integral of eq (18), we get the result as

$$S_E(x) = -2\log\mu - \log\pi + \log\mu\left(\frac{1+\pi}{\pi}\right)\log\left(-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\right)k\left[E_i(z) + \log\left(\frac{-\mu}{z}\right)e^z\right]\Big|_0^\infty$$

$$-\sum_{k=1}^{\infty}\frac{(-1)^k}{k}\mu^k(-1)^{2k}\frac{}{)(1-k,-z)}\Big|_0^\infty - (z-1)e^z + E_i(z) + \log\left(\frac{-\mu}{z}\right)e^z$$

Where, $E_i$is the exponential integral. Further, it was assuming that $k-2 \neq 0,\ k-1 \neq 0$ and $)(.)$ is a gamma function.

7. Real Data Application

This section utilized the COVID-19 and repair time data sets to evaluate the robustness of the proposed distribution. Further, the proposed distribution is assessed against the Weibull (Wb), Transmuted Inverse Weibull (TIW), Inverse Weibull (IW), Alpha Power Weibull (APW) distributions and Exponentiated Inverse Flexible Weibull Extension (EIFWE) distribution for the stability of proposed distribution. For further evaluation the effectiveness of the proposed model, AIC, CAIC, HQIC, and BIC are utilized.

Data Set 1: Mortality Rate of the COVID-19 Patients from Mexico

The dataset has 106 observations showing the patients mortality rate during the COVID-19 pandemic in Mexico is follows as.

1.7652, 1.2210, 1.8782, 2.9924, 2.0766, 1.4534, 2.6440, 3.2996, 2.3330, 1.2030, 2.1710, 1.2244, 1.3312, 0.6880, 1.1708, 2.1370, 2.0070, 1.0484, 0.8688, 1.0286, 1.5260, 2.9208, 1.5806, 1.2740, 0.7074, 1.2654, 0.9460, 0.6430, 1.8568, 2.5756, 1.7626, 2.0086, 1.4520, 1.1970, 1.2824, 0.6790, 0.8848, 1.9870, 1.5680, 1.9100, 0.6998, 0.7502, 1.3936, 0.6572, 2.0316, 1.6216, 1.3394, 1.4302, 1.3120, 0.4154, 0.7556, 0.5976, 0.6672, 1.3628, 1.6650, 1.5708, 1.7102, 0.6456, 1.4972, 1.3250, 1.2280, 0.9818, 0.9322, 1.0784, 2.4084, 1.7392, 0.3630, 0.6654, 1.0812, 1.2364, 0.2082, 0.3600, 0.9898, 0.8178, 0.6718, 0.4140, 0.6596, 1.0634, 1.0884, 0.9114, 0.8584, 0.5000, 1.3070, 0.9296, 0.9394, 1.0918, 0.8240, 0.7844, 0.6438, 0.2804, 0.4876, 0.6514, 0.7264, 0.6466, 0.6054, 0.4704, 0.2410, 0.6436, 0.5852, 0.5202, 0.4130, 0.6058, 0.4116, 0.4652, 0.5012 and 0.3846.

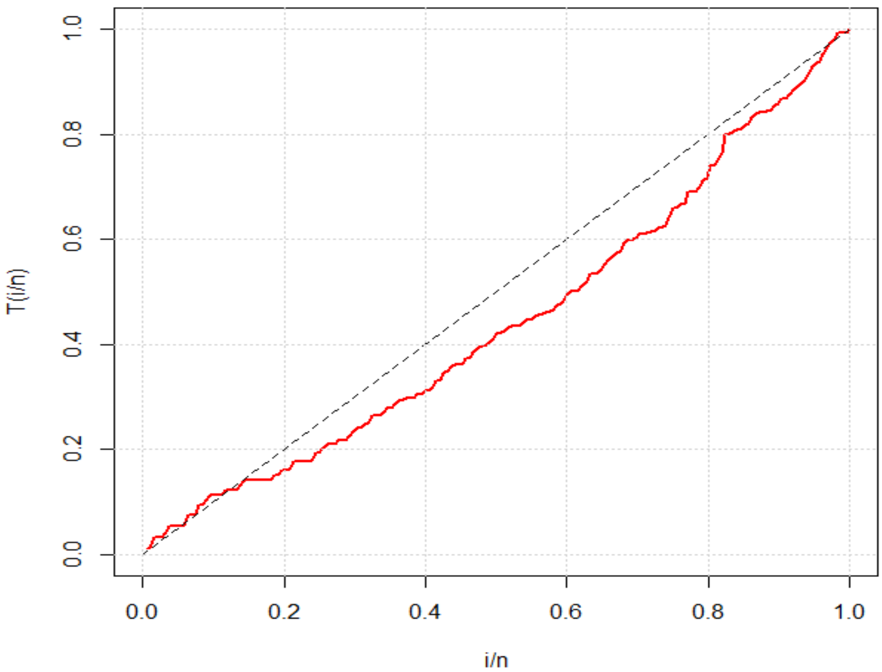

Fig-3: TTT plot of COVID-19 Data

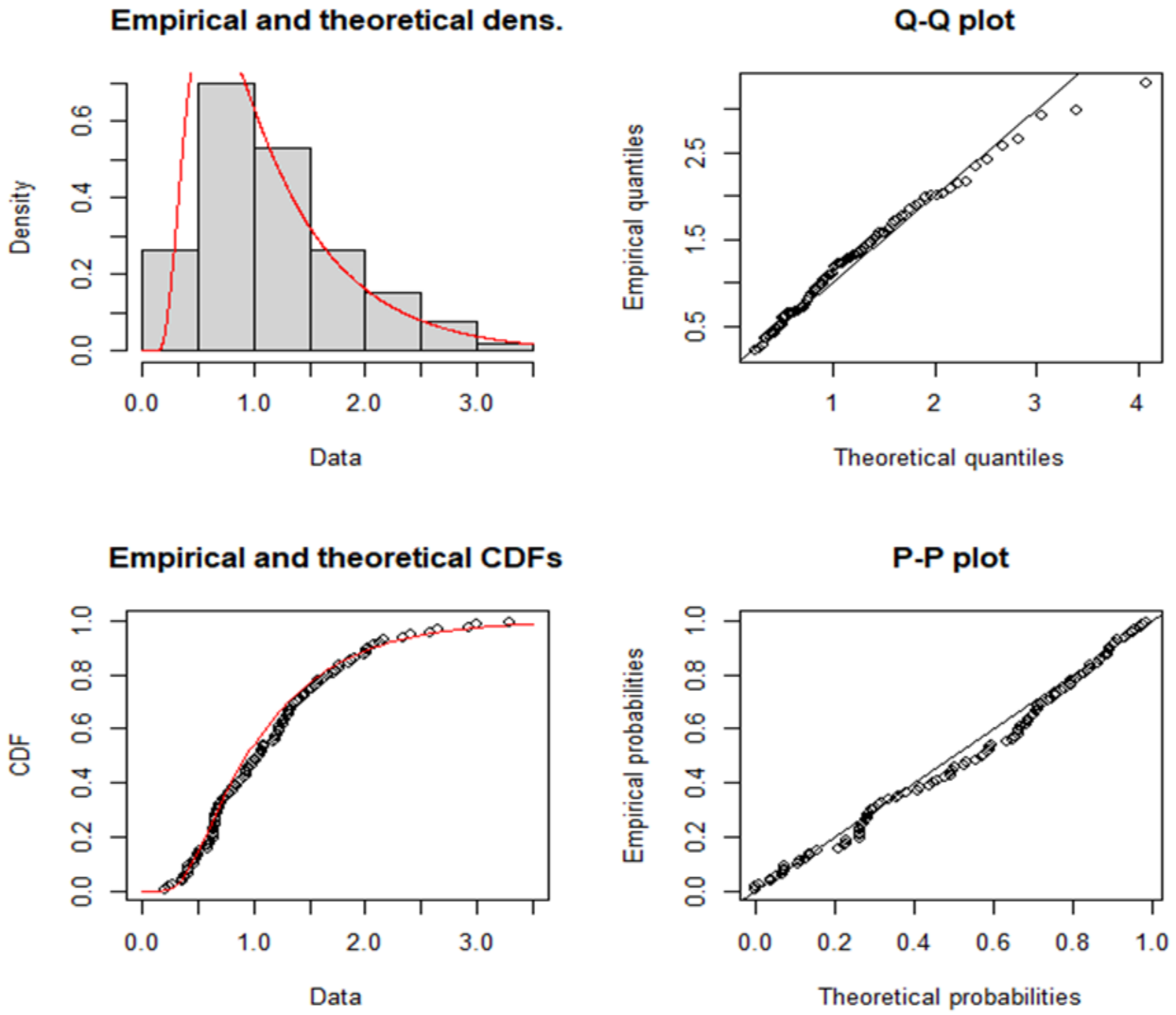

Fig-4: Theoretical, Empirical, QQ & PP Plots of COVID-19 Data

Table 2: Maximum Likelihood Estimates	
Model	**Estimates**
ETW	0.8946919, 1.2616699
Wb	0.5859061, 1.9205049
TIW	1.340692, 1.234306
IW	1.1682595, 0.8635808
APW	0.2411186, 2.1283381, 0.3892093

Table 3: Goodness of Fit Criteria				
Model	**AIC**	**CAIC**	**BIC**	**HQIC**
ETW	188.7951	188.9116	194.122	190.9541
Wb	191.3865	191.503	196.7134	193.5455
TIW	199.9303	200.0469	205.2572	202.0894
IW	245.9029	246.0194	251.2298	248.0619
APW	191.304	191.5393	199.2943	194.5425

Fig-3. the TTT plot for COVID-19 data which noticeably reveals a non-monotonic hazard rate shape, while Fig 4 demonstrated the theoretical and empirical with P-P and Q-Q plots for data set confirmations the best fit of the data on ETW distribution. Table 2 depicts the MLEs for unknown parameters of ETW distribution. Likewise, Table 3 offers the results of various criteria intended for the selection of statistical models. The findings of the goodness of fit test are under Table-3, indicates that ETW better fits than other well-known distribution.

Data Set 2: Airborne Communication Transceiver Repair Time

The data set is comprised of maintenance repair times for an airborne communication transceiver with the values 0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 2.7, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, and 24.5.

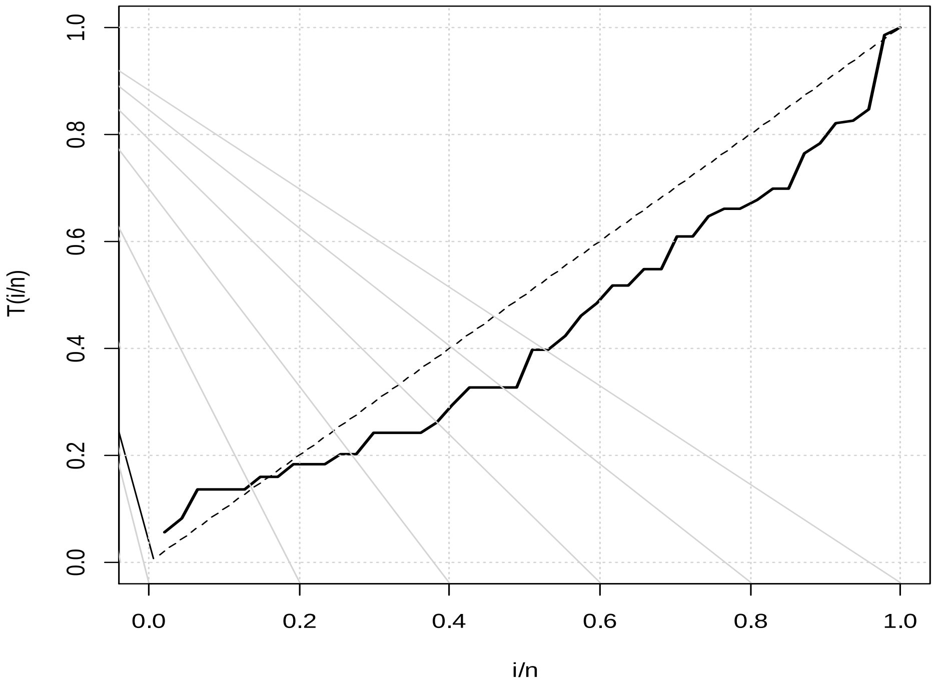

Fig-5: TTT plot for Airborne Communication Transceiver Repair Time

Fig-5. tell us how well the model fits the data. The convex graph in Fig-5 clearly illustrates that this data indicates the presence of a non-monotonic hazard rate shape.

Table 5: Maximum Likelihood Estimates

Models	Estimates
ETW	0.09057463, 0.96825760
W	0.3250289, 0.9094521
APW	0.03415170, 1.10032953, 0.09698703
EIFWE	0.3874405, 0.2601615, 1.0323949
TIW	-0.2843674, 1.0527772

Table 6: Goodness of Fit Criteria

Models	AIC	CAIC	BIC	HQIC
ETW	210.885	211.1577	214.5852	212.2774
W	218.3281	218.6016	222.0291	219.7213
APW	216.4851	217.0432	222.0355	218.5737
EIFWE	215.0345	215.5927	220.585	217.1232
TIW	211.3585	211.6312	215.0588	212.7509

In Table-5, the MLE's for the ETW distribution's unknown parameters are given. The results of the goodness of fit measures are given Table-6. The results clearly show a fever value of all these criteria for the proposed model as compared to the existing models. Thus, the addition of this new probability distribution to the existing literature of probability theory plays a prominent role in further improvement of the model.

8. Simulations

In this section, equation (13) is employed to produce artificial data from the ETW distribution which is iterated 1000 times for different values of parameter with varying sample size. Table 7 illustrates that with increasing sample size, the Mean Square Error (MSE) and bias decrease, implying that the proposed model outperforms. Mathematically, MSE and Bias are defined as.

$$MSE = \frac{1}{K}\sum_{i=1}^{K}(\beta_i - \beta)^2$$

$$Bias = \frac{1}{K}\sum_{i=1}^{K}(\beta_i - \beta)$$

Table 7: MSE and Bias of ETW Distribution

a	b	n	$MSE\ (\mu)$	$MSE\ (\pi)$	$Bias\ (\mu)$	$Bias\ (\pi)$
0.02	0.2	30	0.019866	0.000893	0.076949	0.001775
		60	0.004828	0.000415	0.034237	0.000822
		90	0.002803	0.000273	0.025201	0.000767
		120	0.001714	0.000202	0.019572	0.000428
		100	0.002406	0.000271	0.024206	0.000153
		200	0.000669	0.000125	0.011555	0.000668
		300	0.000502	8.90E-05	0.009094	0.0007
0.3	0.03	30	0.043341	2.82E-05	0.040909	0.001533
		60	0.024179	1.43E-05	0.013737	0.001012
		90	0.016235	8.46E-06	0.012569	0.000648
		100	0.01449	7.44E-06	0.011915	0.000453
		200	0.006738	3.56E-06	0.007297	0.000268
		300	0.00503	2.40E-06	0.005076	0.000142
0.2	0.02	30	0.038175	1.25E-05	0.058581	0.001009
		60	0.016242	5.96E-06	0.026375	0.000523
		90	0.011133	3.77E-06	0.015124	0.000384
		100	0.010623	3.49E-06	0.019473	0.000237
		200	0.005172	1.73E-06	0.008484	0.000216
		300	0.003298	1.12E-06	0.006335	0.000119

9. Conclusion

Using the FETW family, we have developed a new lifetime distribution called ETW distribution with two parameters. Various statistical features of the ETW distribution are computed i.e., order statistics, moments, quantile function, hazard function, survival function, and mean residual life function. For estimation of unknown parameters, we used the traditional method maximum likelihood. We have tested both simulated data and real data of the mortality rate COVID-19 patients of Mexico and Airborne communication transceiver repair time to examine the effectiveness of the proposed model. The proposed distribution is put up against a number of well-known distributions, including the Weibull distribution. Results from the COVID-19 and repair time data sets reveal that the ETW distribution is a better fit and more flexible compared to other distributions.

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