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# Development Of Efficient Estimators For Finite Population Mean Using Dual Auxiliary Variables In Two Phase Sampling

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**Abstract**
The current article aims at the development of some new enhanced estimators for the finite population mean under the scenario of two phase sampling scheme. Two families of estimators have been developed and their first order approximated expressions were derived for Bias and Mean Square Error (MSE). We also determined the situations where the new estimator families demonstrate greater performance relative to the existing estimators. The efficiency of the estimators was assessed through real and artificial data sets by comparing their MSEs and Percentage Relative Efficiencies (PRE). In the end, we concluded that the new families of estimators perform better than all the estimators analyzed in this study.

**Keywords:** Auxiliary variable, Efficiency, Precision, Ratio estimator, Two phase sampling.

## 1. Introduction

It is a fact that adding auxiliary variables that are highly correlated with study variable "y" boosts the precision of the estimator of mean. Different estimators are to be adopted dependent upon scenarios where applied. Traditional single-phase sampling is more appropriate when auxiliary material is readily accessible for the entire population and the budget of collecting data for the study variable is reasonable. But in case of lacking prior information of auxiliary variable the survey becomes unaffordable for this purpose. Therefore, two-phase sampling, commonly called double sampling, is ideal for estimating the auxiliary variables from sample data. this involves taking a large initial sample to compute auxiliary variables, followed by selecting a small second phase sample from this initial large sample.
Two-phase sampling introduced by Neyman (1938) for stratification is a useful technique for improving estimation precision. This method involves a sampling design where the nature and size of the units remains consistent across both phases. This method is particularly useful when different items required different quantity of elements to achieve the anticipated precision. This technique utilizes the information obtained from the initial sample to enhance the precision of the data observed in the secondary sample (Neyman, 1992).
In double sampling, both regression and ratio estimation procedures are employed for estimating the finite population mean. The ratio estimator uses previously gathered information closely related to the study variable, whereas the regression estimator is used when there is a linear relationship exists between the study variable and the auxiliary variables. Generally regression estimator is preferred over ratio estimator unless the regression line pass through the origin in which case both the estimators are similarly significant, and the choice between them is based on judgment. (Hamad et al 2013). Recently many estimators been developed some in ratio form (Kadilar & Cingi 2004,2006, Shabbir & Gupta 2005, Singh & Espejo 2007, Thongsak & Lawson 2021, Muhammad et al. 2023, Wang et al. 2023), others in ratio exponential (Viswakarma & Gangele 2014, Sanaullah et al. 2018, Oyeyeme et al. 2023, Bushan and Kumar 2023, Daraz et al. 2024) and many in ratio cum product exponential form (Kung'u & Nderitu 2016, Verma et al. 2023, Subzar et al. 2023, Sher et al. 2024, Zaagan et al. 2024).

## 2. Methodology

Let the population be composed of N units, and $y_i$, $x_i$ and $z_i$ $\left(i=1,2,....,N\right)$ denote the values of the i-th unit for the variables Y, X and Z respectively. Here $y$ is the variable of interest, $x$ is primary auxiliary variable that has high correlation with $y$ and $z$ is the secondary auxiliary variable that has less correlation with $y$ compared to $x$. First, $n_1$ are drawn as an initial sample from the overall N units and the sample means of auxiliary variables $\bar{x}_1 = \sum_{i=1}^{n_1} x_i$ $\bar{z}_1 = \sum_{i=1}^{n_1} z_i$ are observed. Let the second phase sample of size $n$ $(n < n_1)$ is taken from this initial sample and the values of $\bar{y} = \sum_{i=1}^{n} y_i$, $\bar{x} = \sum_{i=1}^{n} x_i$ and $\bar{z} = \sum_{i=1}^{n} z_i$ are obtained. Both the samples are drawn according to a simple random sampling without replacement (SRSWOR) scheme. The notations to be used are:

$\xi_0 = \frac{\bar{y}-\bar{Y}}{\bar{Y}}$, $\xi_1 = \frac{\bar{x}-\bar{X}}{\bar{X}}$, $\xi_2 = \frac{\bar{z}-\bar{Z}}{\bar{Z}}$, $\xi_3 = \frac{\bar{x}_1-\bar{X}}{\bar{X}}$ and $\xi_4 = \frac{\bar{z}_1-\bar{Z}}{\bar{Z}}$

$\frac{\bar{y}}{\bar{Y}} = (1+\xi_0)$, $\frac{\bar{x}}{\bar{X}} = (1+\xi_1)$, $\frac{\bar{z}}{\bar{Z}} = (1+\xi_2)$, $\frac{\bar{x}_1}{\bar{X}} = (1+\xi_3)$, $\frac{\bar{z}_1}{\bar{Z}} = (1+\xi_4)$, so that $E(\xi_0) = E(\xi_1) = E(\xi_2) = E(\xi_3) = E(\xi_4) = 0$, $E(\xi_0^2) = \lambda C_y^2$, $E(\xi_1^2) = \lambda C_x^2$, $E(\xi_2^2) = \lambda C_z^2$, $E(\xi_3^2) = \lambda_1 C_x^2$ $E(\xi_4^2) = \lambda_1 C_z^2$ and $E(\xi_0\xi_1) = \lambda C_{yx}$, $E(\xi_0\xi_2) = \lambda C_{yz}$, $E(\xi_0\xi_3) = \lambda_1 C_{yx}$, $E(\xi_0\xi_4) = \lambda_1 C_{yz}$, $E(\xi_1\xi_2) = \lambda C_{xz}$, $E(\xi_1\xi_3) = \lambda_1 C_x^2$, $E(\xi_1\xi_4) = \lambda_1 C_{xz}$, $E(\xi_2\xi_3) = \lambda_1 C_{xz}$, $E(\xi_2\xi_4) = \lambda_1 C_z^2$, $E(\xi_3\xi_4) = \lambda_1 C_{xz}$

where $\lambda = \frac{N-n}{Nn}$, $\lambda_1 = \frac{N-n_1}{n_1 N}$, $\lambda_2 = \frac{1}{n} - \frac{1}{n_1} = \lambda - \lambda_1$.

The classical estimator of mean is defined by $t_0 = \bar{y}$ and its variance is given as $V(t_0) = \bar{Y}^2 \lambda C_y^2$

The ratio estimator of mean proposed by Kumar and Bahl (2006) in double sampling is shown as

$$t_r = \bar{y}\left(\frac{\bar{x}_1}{\bar{x}}\right) \tag{1}$$

The first order approximated MSE of their suggested estimator is given as

$$MSE(t_r) = \bar{Y}^2\left(\lambda C_y^2 + \lambda_2 C_x^2 (1-2\psi)\right) \tag{2}$$

where $\psi = \rho_{yx}\frac{C_y}{C_x}$

The dual to product estimator by Singh and Choudhury (2012) in double sampling is given as

$$t_P = \bar{y}\left(\frac{\bar{x}}{\bar{x}_1}\right) \tag{3}$$

The first order approximated MSE of their estimator is shown as

$$MSE(t_P) = \bar{Y}^2\left(\lambda C_y^2 + \lambda_2 C_x^2 (1+2\psi)\right) \tag{4}$$

Raj (1965) introduced the following estimator in case of several variates in two phase sampling

$$t_{Raj} = w\left\{\bar{y} + b_{yx}(\bar{x}_1 - \bar{x})\right\} + (1-w)\left\{\bar{y} + b_{yz}(\bar{z}_1 - \bar{z})\right\} \tag{5}$$

Where w is a constant to be suitably chosen, $b_{yx} = \frac{s_{yx}}{s_x^2}$ and $b_{yz} = \frac{s_{yz}}{s_z^2}$ are the sample counter parts of the population regression coefficients. The MSE of the Raj's estimator is given as

$$MSE(t_{Raj}) = \lambda \bar{Y}^2 C_y^2 + \lambda_2 \bar{Y}^2 C_y^2 \left[1 - \rho_{yx}^2 - \frac{\rho_{yz}^2(\rho_{yz} - \rho_{yx}\rho_{xz})^2}{\rho_{yx}^2 + \rho_{yz}^2 - 2\rho_{yx}\rho_{yz}\rho_{xz}}\right] \tag{6}$$

Mohanty (1967) combined the ratio and regression estimator to propose an estimator for population mean in double sampling,

$$t_M = \left\{\bar{y} - b_{yx}(\bar{x}_1 - \bar{x})\right\}\frac{\bar{z}}{\bar{z}_1} \tag{7}$$

The expression for MSE of Raj's estimator is listed as

$$MSE\left(t_M\right)=\bar{Y}^2\left[\lambda C_y^2+\lambda_2\left\{\rho_{xz}^2 C_z^2-\left(\rho_{yx}C_y-\rho_{xz}C_z\right)^2+\left(C_z-\rho_{yz}C_y\right)^2-C_y^2\rho_{yz}^2\right\}\right] \tag{8}$$

The Mukerjee et al (1987) Regression estimator is written as

$$t_{Muk}=\bar{y}+b_{yx}\left(\bar{x}_1-\bar{x}\right)+b_{yz}\left(\bar{z}_1-\bar{z}\right) \tag{9}$$

MSE of the estimator is shown as

$$MSE\left(t_{Muk}\right)=\bar{Y}^2C_y^2\left[\lambda-\lambda_2\left\{\rho_{yx}^2+\rho_{yz}^2-2\rho_{yx}\rho_{yz}\rho_{xz}\right\}\right] \tag{10}$$

Hanif et al (2010) proposed the regression ratio-product estimator in double sampling in following form

$$t_{Han}=\left\{\bar{y}+b_{yx}\left(\bar{x}_1-\bar{x}\right)\right\}\left\{w_1\frac{\bar{z}_1}{\bar{z}}+\left(1+w_1\right)\frac{\bar{z}}{\bar{z}_1}\right\} \tag{11}$$

$$MSE\left(t_{Han}\right)=\bar{Y}^2C_y^2\left[\lambda-\lambda_2\left\{\rho_{yx}^2+\left(\rho_{yz}-\rho_{xy}\rho_{xz}\right)^2\right\}\right] \tag{12}$$

Khare et al (2013) introduced the following regression ratio type estimator in double sampling with dual auxiliaries.

$$t_{kh_1}=\bar{y}+b_{yx}\left\{\bar{x}_1\left(\frac{\bar{z}_1}{\bar{z}}\right)^{\alpha}-\bar{x}\right\} \tag{13}$$

Where $\alpha$ is a constant. For optimum value of $\alpha=\dfrac{\rho_{yx}C_x}{\rho_{yz}C_z}$ the minimum MSE is

$$MSE\left(t_{kh_1}\right)=\bar{Y}^2C_y^2\left[\lambda-\lambda_2\rho_{yx}^2-\lambda_1\rho_{yz}^2\right] \tag{14}$$

## Proposed Estimator

### i. First proposed estimator

Motivated by Muneer et al (2017) and Shabbir et al (2021) we propose the following class of estimator for population mean

$$t_{\mathrm{Pro1}}=\left[k_1\bar{y}+k_2\right]\left[\omega\frac{\bar{x}_1}{\bar{x}}\left\{2-\exp\left[\frac{u\left(\bar{z}-\bar{z}_1\right)}{u\left(\bar{z}+\bar{z}_1\right)+2v}\right]\right\}+\left(1-\omega\right)\frac{\bar{x}}{\bar{x}_1}\exp\left[\frac{u\left(\bar{z}_1-\bar{z}\right)}{u\left(\bar{z}_1+\bar{z}\right)+2v}\right]\right] \tag{15}$$

Various estimators can be derived from the given class by assigning diverse values to *u, v* and ω. Here $k_1$ and $k_2$ are constants that minimize MSE, and their values are obtained by minimizing process. *u, v* and *ω* can take on any suitable value or known parameters of the auxiliary variables as they are general constants. To find the MSE expression of the estimator, we can rewrite equation (15) in terms of the error as shown below

$$t_{\mathrm{Pro1}}=\left[k_1\bar{Y}\left(1+\xi_0\right)+k_2\right]\left[\begin{array}{l}\omega\left(1+\xi_2\right)\left(1+\xi_1\right)^{-1}\left\{2-\exp\left[\dfrac{u\bar{X}\left(\xi_3-\xi_4\right)}{u\bar{X}\left(\xi_3+\xi_4\right)+2\left(u\bar{X}+v\right)}\right]\right\}\\+\left(1-\omega\right)\left(1+\xi_1\right)\left(1+\xi_2\right)^{-1}\exp\left[\dfrac{u\bar{X}\left(\xi_4-\xi_3\right)}{u\bar{X}\left(\xi_4+\xi_3\right)+2\left(u\bar{X}+v\right)}\right]\end{array}\right] \tag{16}$$

After applying Taylor and exponential series to Eq. (16) we have

$$t_{\mathrm{Pro1}}=\left[k_1\bar{Y}\left(1+\xi_0\right)+k_2\right]\left[\begin{array}{l}1+\vartheta_1\xi_1-\vartheta_1\xi_2-\varepsilon_1\varepsilon_2+\omega\xi_1^2+\vartheta_2\xi_2^2-\dfrac{\eta}{2}\xi_3+\dfrac{\eta}{2}\xi_4-\vartheta_3\xi_1\xi_3\\+\vartheta_3\xi_1\xi_4+\vartheta_3\xi_2\xi_3-\vartheta_3\xi_2\xi_4+\vartheta_4\xi_3^2-\vartheta_5\xi_4^2-\vartheta_6\xi_3\xi_4\end{array}\right] \tag{17}$$

Where $\eta=\dfrac{u\bar{Z}}{u\bar{Z}+v}$,

$\vartheta_1=1-2\omega$, $\vartheta_2=1-\omega$, $\vartheta_3=\left(\dfrac{1}{2}-\omega\right)\eta$, $\vartheta_4=\left(\dfrac{3-2\omega}{8}\right)\eta^2$, $\vartheta_5=\left(\dfrac{1+2\omega}{8}\right)\eta^2$ and $\vartheta_6=\left(\dfrac{1-2\omega}{4}\right)\eta^2$

The difference equation is given below in Eq. (18)

$$t_{\mathrm{Pro1}}-\bar{Y}=k_1\bar{Y}\left(\begin{array}{l}1+\xi_0+\vartheta_1\xi_1+\vartheta_1\xi_0\xi_1-\vartheta_1\xi_2-\vartheta_1\xi_0\xi_2-\xi_1\xi_2\\+\omega\xi_1^2+\vartheta_2\xi_2^2-\dfrac{\eta}{2}\xi_3-\dfrac{\eta}{2}\xi_0\xi_3+\dfrac{\eta}{2}\xi_4+\dfrac{\eta}{2}\xi_0\xi_4-\vartheta_3\xi_1\xi_3\\+\vartheta_3\xi_1\xi_4+\vartheta_3\xi_2\xi_3-\vartheta_3\xi_2\xi_4+\vartheta_4\xi_3^2-\vartheta_5\xi_4^2-\vartheta_6\xi_3\xi_4\end{array}\right)+k_2\left(\begin{array}{l}1+\vartheta_1\xi_1-\vartheta_1\xi_2-\xi_1\xi_2+\omega\xi_1^2+\vartheta_2\xi_2^2\\-\dfrac{\eta}{2}\xi_3+\dfrac{\eta}{2}\xi_4-\vartheta_3\xi_1\xi_3+\vartheta_3\xi_1\xi_4+\vartheta_3\xi_2\xi_3\\-\vartheta_3\xi_2\xi_4+\vartheta_4\xi_3^2-\vartheta_5\xi_4^2-\vartheta_6\xi_3\xi_4\end{array}\right)-\bar{Y} \tag{18}$$

Bias of the estimator is obtained through taking expectation on both sides of the Eq. (18) as

$$Bias\left(t_{\text{Pro1}}\right)=k_1\bar{Y}\begin{pmatrix}1+\vartheta_1\left(\lambda-\lambda_1\right)C_{yx}-\left(\lambda_1-\omega\lambda-\vartheta_2\lambda_1\right)C_x^2-\frac{\eta}{2}\left(\lambda-\lambda_1\right)C_{yz}\\-\vartheta_3\left(\lambda-\lambda_1\right)C_{xz}+\left(\vartheta_4\lambda-\vartheta_5\lambda_1-\vartheta_6\lambda_1\right)C_z^2\end{pmatrix}+k_2\begin{pmatrix}1-\left(\lambda_1-\omega\lambda-\vartheta_2\lambda_1\right)C_x^2-\vartheta_3\left(\lambda-\lambda_1\right)C_{xz}\\+\left(\vartheta_4\varepsilon\lambda-\vartheta_5\lambda_1-\vartheta_6\lambda_1\right)C_z^2\end{pmatrix}-\bar{Y} \qquad (19)$$

Now by after taking square Eq. (18) and applying expectation we get the following MSE expression

$$\begin{aligned}MSE\left(t_{\text{Pro1}}\right)&=k_1^2\bar{Y}^2\left\{1+\lambda C_y^2+\left(\Delta_1\lambda+\Delta_2\lambda_1-\Delta_5\lambda_1\right)C_x^2+\left(\Delta_3\lambda-\Delta_4\lambda_1-\Delta_7\lambda_1\right)C_z^2+4\vartheta_1\left(\lambda-\lambda_1\right)C_{yz}-2\eta\left(\lambda-\lambda_1\right)C_{yz}-\Delta_6\left(\lambda-\lambda_1\right)C_{xz}\right\}\\&+k_2^2\left\{1+\left(\Delta_1\lambda+\Delta_2\lambda_1-\Delta_5\lambda_1\right)C_x^2+\left(\Delta_3\lambda-\Delta_4\lambda_1-\Delta_7\lambda_1\right)C_z^2-\Delta_6\left(\lambda-\lambda_1\right)C_{xz}\right\}\\&-2k_1\bar{Y}^2\left\{1+\left(\omega\lambda+\vartheta_2\lambda_1-\lambda_1\right)C_x^2+\left(\vartheta_4\lambda-\vartheta_5\lambda_1-\vartheta_6\lambda_1\right)C_z^2+\vartheta_1\left(\lambda-\lambda_1\right)C_{yx}-\frac{\eta}{2}\left(\lambda-\lambda_1\right)C_{yz}-\vartheta_3\left(\lambda-\lambda_1\right)C_{xz}\right\}\\&-2k_2\bar{Y}\left\{1+\left(\omega\lambda+\vartheta_2\lambda_1-\lambda_1\right)C_x^2+\left(\vartheta_4\lambda-\vartheta_5\lambda_1-\vartheta_6\lambda_1\right)C_z^2-\vartheta_3\left(\lambda-\vartheta_3\lambda_1\right)C_{xz}\right\}\\&+2k_1k_2\bar{Y}\left\{1+\left(\Delta_1\lambda+\Delta_2\lambda_1-\Delta_5\lambda_1\right)C_x^2+\left(\Delta_3\lambda-\Delta_4\lambda_1-\Delta_7\lambda_1\right)C_z^2+2\vartheta_1\left(\lambda-\lambda_1\right)C_{yx}-\eta\left(\lambda-\lambda_1\right)C_{yz}-\Delta_6\left(\lambda-\lambda_1\right)C_{xz}\right\}+\bar{Y}^2\end{aligned}$$

(20)

Eq. (20) is the MSE of the first proposed class where $\Delta_1=\vartheta_1^2+2\omega$, $\Delta_2=\vartheta_1^2+2\vartheta_2$, $\Delta_3=\left(2\vartheta_4+\frac{\eta^2}{4}\right)$, $\Delta_4=\left(2\vartheta_5-\frac{\eta^2}{4}\right)$, $\Delta_5=2\left(\vartheta_1^2+1\right)$, $\Delta_6=\left(2\vartheta_3+\frac{\eta}{2}\vartheta_1\right)$ and $\Delta_7=\left(2\vartheta_6+\frac{\eta^2}{2}\right)$

To find the values of $k_1$ and $k_2$, we convert the above Eq. (20) to below simplified Eq. (21)

$$MSE\left(t_{\text{Pro1}}\right)=k_1^2\bar{Y}^2A_{pr}+k_2^2B_{pr}-2k_1\bar{Y}^2C_{pr}-2k_2\bar{Y}D_{pr}+2k_1k_2\bar{Y}E_{pr}+\bar{Y}^2 \qquad (21)$$

Where

$A_{pr}=1+\lambda C_y^2+\left(\Delta_1\lambda+\Delta_2\lambda_1-\Delta_5\lambda_1\right)C_x^2+\left(\Delta_3\lambda-\Delta_4\lambda_1-\Delta_7\lambda_1\right)C_z^2+4\vartheta_1\left(\lambda-\lambda_1\right)C_{yz}-2\eta\left(\lambda-\lambda_1\right)C_{yz}-\Delta_6\left(\lambda-\lambda_1\right)C_{xz}$,

$B_{pr}=1+\left(\Delta_1\lambda+\Delta_2\lambda_1-\Delta_5\lambda_1\right)C_x^2+\left(\Delta_3\lambda-\Delta_4\lambda_1-\Delta_7\lambda_1\right)C_z^2-\Delta_6\left(\lambda-\lambda_1\right)C_{xz}$,

$C_{pr}=1+\left(\omega\lambda+\vartheta_2\lambda_1-\lambda_1\right)C_x^2+\left(\vartheta_4\lambda-\vartheta_5\lambda_1-\vartheta_6\lambda_1\right)C_z^2+\vartheta_1\left(\lambda-\lambda_1\right)C_{yx}-\frac{\eta}{2}\left(\lambda-\lambda_1\right)C_{yz}-\vartheta_3\left(\lambda-\lambda_1\right)C_{xz}$,

$D_{pr}=1+\left(\omega\lambda+\vartheta_2\lambda_1-\lambda_1\right)C_x^2+\left(\vartheta_4\lambda-\vartheta_5\lambda_1-\vartheta_6\lambda_1\right)C_z^2-\vartheta_3\left(\lambda-\vartheta_3\lambda_1\right)C_{xz}$ and

$E_{pr}=1+\left(\Delta_1\lambda+\Delta_2\lambda_1-\Delta_5\lambda_1\right)C_x^2+\left(\Delta_3\lambda-\Delta_4\lambda_1-\Delta_7\lambda_1\right)C_z^2+2\vartheta_1\left(\lambda-\lambda_1\right)C_{yx}-\eta\left(\lambda-\lambda_1\right)C_{yz}-\Delta_6\left(\lambda-\lambda_1\right)C_{xz}$

Now differentiating Eq. (21) w.r.t. $k_1$ and $k_2$, and equating to zero we get the following two equations

$$\frac{\partial MSE\left(t_{pro1}\right)}{\partial k_1}=0 \text{ and } \frac{\partial MSE\left(t_{pro1}\right)}{\partial k_2}=0$$

So we obtain

$$k_1\bar{Y}^2A_{pr}+k_2\bar{Y}E_{pr}-\bar{Y}^2C_{pr}=0 \qquad (22)$$

$$k_1\bar{Y}E_{pr}+k_2B_{pr}-\bar{Y}D_{pr}=0 \qquad (23)$$

Solving both Eq. (22) & Eq. (23) simultaneously we obtain the following optimum values of $k_1=\frac{B_{pr}C_{pr}-D_{pr}E_{pr}}{A_{pr}B_{pr}-E_{pr}^2}$ and $k_2=\frac{\bar{Y}\left(A_{pr}D_{pr}-C_{pr}E_{pr}\right)}{A_{pr}B_{pr}-E_{pr}^2}$,

using these values, the estimator's minimum MSE is given by the following form

$$MSE\left(t_{\text{Pro1}}\right)_{\min}\cong\bar{Y}^2\left\{1-\frac{A_{pr}D_{pr}^2+B_{pr}C_{pr}^2-2C_{pr}D_{pr}E_{pr}}{A_{pr}B_{pr}-E_{pr}^2}\right\} \qquad (24)$$

Eq. (24) will gives the optimum values of the MSE for the first suggested class of mean estimator.

**ii. Second Proposed Estimator**

Motivated by Hanif et al (2010) and Shabbir et al (2021), we proposed the following exponential ratio cum product estimator in two phase random sampling with two auxiliaries.

$$t_{pro2}=\left(k_3\bar{y}+k_4\right)\left[\omega\frac{\bar{x}_1}{\bar{x}}+\left(1-\omega\right)\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{u\left(\bar{z}_1-\bar{z}\right)}{u\left(\bar{z}_1+\bar{z}\right)+2v}\right] \quad (25)$$

Where $k_1$ and $k_2$ are the optimizing constants, whose values are to be determined. omega ($\omega$) can take any value in the range (0, 1) and the $u$ and $v$ are the general constants that can assume any value from the known parameters, like $C_x$, $C_z$, $\rho_{yx}$, $\rho_{yz}$, $\rho_{xz}$, $\beta_x$, $\beta_z$ etc.

**Table 1:** Shows few of the deduced estimators of mean from the suggested classes.

| S. No | $u$ | $v$ | $\omega$ | First suggested class of Estimators | Second suggested class of Estimators |
|---|---|---|---|---|---|
| 1 | 1 | 0 | 1 | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\left[\frac{\bar{x}_1}{\bar{x}}\right]\left\{2-\exp\left[\frac{\bar{z}-\bar{z}_1}{\bar{z}_1+\bar{z}}\right]\right\}$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}_1}{\bar{x}}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\bar{z}_1+\bar{z}}\right]$ |
| 2 | $\rho_{yz}$ | 1 | 1 | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\left[\frac{\bar{x}_1}{\bar{x}}\right]\left\{2-\exp\left[\frac{\rho_{yz}\left(\bar{z}-\bar{z}_1\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2}\right]\right\}$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}_1}{\bar{x}}\right]\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2}\right]$ |
| 3 | 1 | $C_z$ | 1 | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\left[\frac{\bar{x}_1}{\bar{x}}\right]\left\{2-\exp\left[\frac{\left(\bar{z}-\bar{z}_1\right)}{\left(\bar{z}_1+\bar{z}\right)+2C_z}\right]\right\}$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}_1}{\bar{x}}\right]\exp\left[\frac{\left(\bar{z}_1-\bar{z}\right)}{\left(\bar{z}_1+\bar{z}\right)+2C_z}\right]$ |
| 4 | 1 | $\beta_z$ | 1 | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\left[\frac{\bar{x}_1}{\bar{x}}\right]\left\{2-\exp\left[\frac{\left(\bar{z}-\bar{z}_1\right)}{\left(\bar{z}_1+\bar{z}\right)+2\beta_z}\right]\right\}$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}_1}{\bar{x}}\right]\exp\left[\frac{\left(\bar{z}_1-\bar{z}\right)}{\left(\bar{z}_1+\bar{z}\right)+2\beta_z}\right]$ |
| 5 | 1 | 0 | 0 | $t_{pro2}=\left[k_1\bar{y}+k_2\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\bar{z}_1+\bar{z}}\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\bar{z}_1+\bar{z}}\right]$ |
| 6 | 1 | 1 | 0 | $t_{pro2}=\left[k_2\bar{y}+k_2\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\left(\bar{z}_1+\bar{z}\right)+2}\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\left(\bar{z}_1+\bar{z}\right)+2}\right]$ |
| 7 | $\rho_{yz}$ | 1 | 0 | $t_{\text{Pro1}}=\left[k_1\bar{y}+k_2\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2}\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\rho\left(\bar{z}_1-\bar{z}\right)}{\rho\left(\bar{z}_1+\bar{z}\right)+2}\right]$ |
| 8 | 1 | $\beta_z$ | 0 | $t_{\text{Pro1}}=\left[k_1\bar{y}+k_2\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\left(\bar{z}_1+\bar{z}\right)+2\beta_z}\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{\bar{x}}{\bar{x}_1}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\left(\bar{z}_1+\bar{z}\right)+2\beta_z}\right]$ |
| 9 | 1 | 0 | ½ | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\frac{1}{2}\left[\frac{\bar{x}_1}{\bar{x}}\left\{2-\exp\left[\frac{\bar{z}-\bar{z}_1}{\bar{z}_1+\bar{z}}\right]\right\}+\frac{\bar{x}}{\bar{x}_1}\exp\left[\frac{\bar{z}_1-\bar{z}}{\bar{z}_1+\bar{z}}\right]\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{1}{2}\left\{\frac{\bar{x}_1}{\bar{x}}+\frac{\bar{x}}{\bar{x}_1}\right\}\right]\exp\left[\frac{\bar{z}_1-\bar{z}}{\bar{z}_1+\bar{z}}\right]$ |
| 10 | $\rho_{yz}$ | 1 | ½ | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\frac{1}{2}\left[\frac{\bar{x}_1}{\bar{x}}\left\{2-\exp\left[\frac{\rho_{yz}\left(\bar{z}-\bar{z}_1\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2}\right]\right\}+\frac{\bar{x}}{\bar{x}_1}\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2}\right]\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{1}{2}\left\{\frac{\bar{x}_1}{\bar{x}}+\frac{\bar{x}}{\bar{x}_1}\right\}\right]\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2}\right]$ |
| 11 | $\rho_{yz}$ | $\beta_z$ | ½ | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\frac{1}{2}\left[\frac{\bar{x}_1}{\bar{x}}\left\{2-\exp\left[\frac{\rho_{yz}\left(\bar{z}-\bar{z}_1\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2\beta_z}\right]\right\}+\frac{\bar{x}}{\bar{x}_1}\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2\beta_z}\right]\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{1}{2}\left\{\frac{\bar{x}_1}{\bar{x}}+\frac{\bar{x}}{\bar{x}_1}\right\}\right]\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2\beta_z}\right]$ |
| 12 | $\rho_{yz}$ | $C_z$ | ½ | $t_{\text{Pro1}}=[k_1\bar{y}+k_2]\frac{1}{2}\left[\frac{\bar{x}_1}{\bar{x}}\left\{2-\exp\left[\frac{\rho_{yz}\left(\bar{z}-\bar{z}_1\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2C_z}\right]\right\}+\frac{\bar{x}}{\bar{x}_1}\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2C_z}\right]\right]$ | $t_{pro2}=\left[k_3\bar{y}+k_4\right]\left[\frac{1}{2}\left\{\frac{\bar{x}_1}{\bar{x}}+\frac{\bar{x}}{\bar{x}_1}\right\}\right]\exp\left[\frac{\rho_{yz}\left(\bar{z}_1-\bar{z}\right)}{\rho_{yz}\left(\bar{z}_1+\bar{z}\right)+2C_z}\right]$ |

Table 1 shows only a bunch of the estimators that are constructed from the suggested classes, many others can also be obtained from the suggested classes apart from the table 1. In the deduced estimators some are already exists in the literature and some are purely novel. For Bias and MSE calculation we proceed in subsequent lines.
The Eq. (24) in the simplified form of errors expressed as

$$t_{pro2}=\left[k_3\bar{Y}\left(1+\xi_0\right)+k_4\right]\left[\omega\left(1+\xi_3\right)\left(1+\xi_1\right)^{-1}+\left(1-\omega\right)\left(1+\xi_1\right)\left(1+\xi_3\right)^{-1}\right]\exp\left[\frac{\eta}{2}\left(\xi_4-\xi_2\right)\left\{1+\frac{\eta}{2}\left(\xi_4+\xi_2\right)\right\}^{-1}\right]$$

(25)

By applying the Taylor and Exponential series Eq. (25) becomes

$$t_{pro2}=\left[k_3\bar{Y}\left(1+\xi_0\right)+k_4\right]\left[1+2\omega+\xi_1-\xi_3-\left(1+2\omega\right)\xi_1\xi_3+\gamma\xi_1^2+\left(1-\omega\right)\xi_3^2\right]\left[1+\frac{\omega}{2}\left(\xi_4-\xi_2\right)-\frac{\omega^2}{4}\left(\xi_4^2-\xi_2^2\right)+\frac{\omega^2}{8}\left(\xi_4^2+\xi_2^2-\xi_2\xi_4\right)\right]$$

(26)

The difference equation is obtained from Eq. (26) as

$$t_{pro2}-\bar{Y}=k_3\bar{Y}\left[2\delta+\xi_1-\delta\theta\xi_2-\xi_3+\delta\theta\xi_4+\omega\xi_1^2+\frac{3}{4}\delta\theta^2\xi_2^2+(1+\omega)\xi_3^2-\frac{1}{4}\delta\theta^2\xi_4^2-\frac{\theta}{2}\xi_1\xi_2-2\delta\xi_1\xi_3+\frac{\theta}{2}\xi_1\xi_4+\frac{\theta}{2}\xi_2\xi_3-\frac{\delta\theta^2}{2}\xi_2\xi_4-\frac{\theta}{2}\xi_3\xi_4\right]$$
$$+k_3\bar{Y}\left[2\delta\xi_0+\xi_0\xi_1-\delta\theta\xi_0\xi_2-\xi_0\xi_3+\delta\theta\xi_0\xi_4\right]$$
$$+k_4\left[2\delta+\xi_1-\delta\theta\xi_2-\xi_3+\delta\theta\xi_4+\omega\xi_1^2+\frac{3}{4}\delta\theta^2\xi_2^2+(1+\omega)\xi_3^2-\frac{1}{4}\delta\theta^2\xi_4^2-\frac{\theta}{2}\xi_1\xi_2-2\delta\xi_1\xi_3+\frac{\theta}{2}\xi_1\xi_4+\frac{\theta}{2}\xi_2\xi_3-\frac{\delta\theta^2}{2}\xi_2\xi_4-\frac{\theta}{2}\xi_3\xi_4\right]-\bar{Y}$$

(27)

Taking expectations on both sides of the Eq. (27), Bias of the proposed estimator is listed in Eq. (28) as

$$Bais\left(t_{pro2}\right)=k_3\bar{Y}\left[2\delta+\left(\omega\lambda+(1+\omega-2\delta)\lambda_1\right)C_x^2+(\lambda-\lambda_1)\left\{\frac{3}{4}\delta\theta^2C_z^2+C_{yx}-\delta\theta C_{yz}-\frac{\theta}{2}C_{xz}\right\}\right]$$
$$+k_4\left[2\delta+\left(\omega\lambda+(1+\omega-2\delta)\lambda_1\right)C_x^2+(\lambda-\lambda_1)\left\{\frac{3}{4}\delta\theta^2C_z^2-\frac{\theta}{2}C_{xz}\right\}\right]-\bar{Y} \quad (28)$$

Now taking square of the equation (27) and simplifying we have Eq. (29) as

$$\left(t_{pro2}-\bar{Y}\right)^2=\bar{Y}^2(k_3-1)^2+k_3^2\bar{Y}^2\left[\begin{array}{l}4\omega^2+4\omega+4\delta^2\xi_0^2+\left(2\delta+4\omega^2\right)\xi_1^2+6\delta\theta^2(\delta+\omega)\xi_2^2+\left(1+4\delta^2+2\delta\right)\xi_3^2+4\delta\xi_0\xi_1-4\delta^2\theta\xi_0\xi_2-4\delta\xi_0\xi_3\\+4\delta^2\theta\xi_0\xi_4-4\delta\theta\xi_1\xi_2-2(1+4\omega\delta+2\delta)\xi_1\xi_3+4\delta\theta\xi_1\xi_4+4\delta\theta\xi_2\xi_3-4\delta^2\theta^2\xi_2\xi_4-4\delta\theta\xi_3\xi_4\end{array}\right]$$
$$+k_4^2\left[4\delta^2+(1+4\delta\omega)\xi_1^2+4\delta^2\theta^2\xi_2^2+(1+4\delta+4\delta\omega)\xi_3^2-4\delta\theta\xi_1\xi_2-\left(2+8\delta^2\right)\xi_1\xi_3+4\delta\theta\xi_1\xi_4+4\delta\theta\xi_2\xi_3-4\delta^2\theta^2\xi_2\xi_4-4\delta\theta\xi_3\xi_4\right]$$
$$+2k_3k_4\bar{Y}\left[\begin{array}{l}4\delta^2+(1+4\delta\omega)\xi_1^2+4\delta^2\theta^2\xi_2^2+(1+4\delta+4\delta\omega)\xi_3^2+4\delta\xi_0\xi_1-4\delta^2\theta\xi_0\xi_2-4\delta\xi_0\xi_3+4\delta^2\theta\xi_0\xi_4\\-4\delta\theta\xi_1\xi_2-\left(1+2\delta+2\delta\theta-4\delta^2\right)\xi_1\xi_3+4\delta\theta\xi_1\xi_4+4\delta\theta\xi_2\xi_3-4\delta^2\theta^2\xi_2\xi_4-4\delta\theta\xi_3\xi_4\end{array}\right]$$
$$-2k_3\bar{Y}^2\left[2\omega+\omega\xi_1^2+\frac{3}{4}\delta\theta^2\xi_2^2+(1+\omega)\xi_3^2-\frac{1}{4}\delta\theta^2\xi_4^2+\xi_0\xi_1-\delta\theta\xi_0\xi_2-\xi_0\xi_3+\delta\theta\xi_0\xi_4-\frac{\theta}{2}\xi_1\xi_2-2\delta\xi_1\xi_3+\frac{\theta}{2}\xi_1\xi_4+\frac{\theta}{2}\xi_2\xi_3-\frac{\delta\theta^2}{2}\xi_2\xi_4-\frac{\theta}{2}\xi_3\xi_4\right]$$
$$-2k_4\bar{Y}\left[2\delta+\omega\xi_1^2+\frac{3}{4}\delta\theta^2\xi_2^2+(1+\omega)\xi_3^2-\frac{1}{4}\delta\theta^2\xi_4^2-\frac{\theta}{2}\xi_1\xi_2-2\delta\xi_1\xi_3+\frac{\theta}{2}\xi_1\xi_4+\frac{\theta}{2}\xi_2\xi_3-\frac{\delta\theta^2}{2}\xi_2\xi_4-\frac{\theta}{2}\xi_3\xi_4\right]$$

(29)

To get the MSE, lets take expectation on both sides of the Eq. (29)

$$MSE\left(t_{pro2}\right)=\bar{Y}^2(k_3-1)^2+k_3^2\bar{Y}^2\left[\begin{array}{l}4\omega^2+4\omega+4\delta^2\lambda C_y^2+\left\{2\delta(\lambda-\lambda_1)+4\omega^2\lambda+\left(4\delta^2-8\omega\delta-1\right)\lambda_1\right\}C_x^2+\left\{2\delta^2\theta^2(3\lambda-2\lambda_1)+6\delta\theta^2\omega\lambda\right\}C_z^2\\+4\delta(\lambda-\lambda_1)C_{yx}-4\delta^2\theta(\lambda-\lambda_1)C_{yz}-4\delta\theta(\lambda-\lambda_1)C_{xz}\end{array}\right]$$
$$+k_4^2\left[4\delta^2+\left\{(1+4\delta\omega)(\lambda-\lambda_1)+\left(4\delta-2+8\delta^2\right)\lambda_1\right\}C_x^2+4\delta^2\theta^2(\lambda-\lambda_1)C_z^2-4\delta\theta(\lambda-\lambda_1)C_{xz}\right] \quad (30)$$
$$+2k_3k_4\bar{Y}\left[4\delta^2+\left\{(1+4\delta\omega)(\lambda-\lambda_1)+\left(2\delta-1-2\delta\theta+4\delta^2\right)\lambda_1\right\}\lambda C_x^2+4\delta^2\theta^2(\lambda-\lambda_1)C_z^2+4\delta(\lambda-\lambda_1)C_{yx}-4\delta^2\theta(\lambda-\lambda_1)C_{yz}-4\delta\theta(\lambda-\lambda_1)C_{xz}\right]$$
$$-2k_3\bar{Y}^2\left[2\omega+\left\{\omega(\lambda+\lambda_1)+(1-2\delta)\lambda_1\right\}C_x^2+\frac{3}{4}\delta\theta^2(\lambda-\lambda_1)C_z^2+(\lambda-\lambda_1)C_{yx}-\delta\theta(\lambda-\lambda_1)C_{yz}-\frac{\theta}{2}(\lambda-\lambda_1)C_{xz}\right]$$
$$-2k_4\bar{Y}\left[2\delta+\left\{\omega(\lambda+\lambda_1)+(1-2\delta)\lambda_1\right\}C_x^2+\frac{3}{4}\delta\theta^2(\lambda-\lambda_1)C_z^2-\frac{\theta}{2}(\lambda-\lambda_1)C_{xz}\right]$$

By substitution of the Eq. (30) takes the following form

$$MSE\left(t_{pro2}\right)=\bar{Y}^2(k_3-1)^2+k_3^2\bar{Y}^2A_p+k_4^2B_p-2k_3\bar{Y}^2C_p-2k_4\bar{Y}D_p+2k_3k_4\bar{Y}E_p \quad (31)$$

Where

$$A_p=4\omega^2+4\omega+4\delta^2\lambda C_y^2+\left\{2\delta\lambda_2+4\omega^2\lambda+\left(4\delta^2-8\omega\delta-1\right)\lambda_1\right\}C_x^2+\left\{2\delta^2\theta^2(3\lambda-2\lambda_1)+6\delta\theta^2\omega\lambda\right\}C_z^2$$
$$+4\delta\lambda_2C_{yx}-4\delta^2\theta\lambda_2C_{yz}-4\delta\theta\lambda_2C_{xz}$$

$$B_p=4\delta^2+\left\{(1+4\delta\omega)\lambda_2+\left(4\delta-2+8\delta^2\right)\lambda_1\right\}C_x^2+4\delta^2\theta^2\lambda_2C_z^2-4\delta\theta\lambda_2C_{xz}$$

$$C_p=2\omega+\left\{\omega(\lambda+\lambda_1)+(1-2\delta)\lambda_1\right\}C_x^2+\frac{3}{4}\delta\theta^2\lambda_2C_z^2+\lambda_2C_{yx}-\delta\theta\lambda_2C_{yz}-\frac{\theta}{2}\lambda_2C_{xz}$$

$$D_p=2\delta+\left\{\omega(\lambda+\lambda_1)+(1-2\delta)\lambda_1\right\}C_x^2+\frac{3}{4}\delta\theta^2\lambda_2C_z^2-\frac{\theta}{2}\lambda_2C_{xz}$$

$$E_p=4\delta^2+\left\{(1+4\delta\omega)\lambda_2+\left(2\delta-1-2\delta\theta+4\delta^2\right)\lambda_1\right\}\lambda C_x^2+4\delta^2\theta^2\lambda_2C_z^2+4\delta\lambda_2C_{yx}-4\delta^2\theta\lambda_2C_{yz}-4\delta\theta\lambda_2C_{xz}$$

Now to find the values of $k_1$ and $k_2$ we differentiate Eq. (31) w. r. t. $k_3$ , $k_4$ and equate it to zero as

$$\frac{\partial MSE\left(t_{pro2}\right)}{\partial k_3}=0 \quad \Rightarrow \bar{Y}^2(k_3-1)+\bar{Y}^2k_3A_p-\bar{Y}^2C_p+\bar{Y}k_4E_p=0 \quad (32)$$

$$\frac{\partial MSE\left(t_{pro2}\right)}{\partial k_4}=0\ \Rightarrow k_4B_p-\bar{Y}D_p+\bar{Y}k_3E_p=0 \quad (33)$$

Solving Eq. (32) and Eq. (33) simultaneously, we have

$$k_3=\frac{B_pC_p-D_pE_p+B_p}{A_pB_p-E_p^2+B_p} \text{ and } k_4=\frac{\bar{Y}\left(A_pD_p-C_pE_p+D_p-E_p\right)}{A_pB_p-E_p^2+B_p}$$

After substituting the values, the minimized MSE of the proposed estimator becomes

$$MSE\left(t_{pro2}\right)\cong\bar{Y}^2\left[1-\frac{\left(A_pD_p^2+B_pC_p^2+2B_pC_p+B_p+D_p^2-2C_pD_pE_p-2D_pE_p\right)}{A_pB_p-E_p^2+B_p}\right] \quad (34)$$

**Theoretical comparison**

The proposed estimator $t_{pro1}$ and $t_{pro2}$ give best result as compared to some of the exiting estimators subject to the following conditions.

**a. Conditions for the First Proposed Estimator**

Cond. (i)

By Eq. (29) and Eq.(2), $MSE\left(t_{pro1}\right)\le MSE\left(t_r\right)$, if

$$\lambda C_y^2+\lambda_2C_x^2(1-2\psi)+\Re_1-1\ge 0 \quad (35)$$

Here $\Re_1=\dfrac{A_{pr}D_{pr}^2+B_{pr}C_{pr}^2-2C_{pr}D_{pr}E_{pr}}{A_{pr}B_{pr}-E_{pr}^2}$

Cond. (ii)

By Eq. (29) and Eq. (4), $MSE\left(t_{pro1}\right)\le MSE\left(t_p\right)$, if

$$\lambda C_y^2+\lambda_2C_x^2(1+2\psi)+\Re_1-1\ge 0 \quad (36)$$

Cond. (iii)

By Eq. (29) and Eq.(6), $MSE\left(t_{pro1}\right)\le MSE\left(t_{Raj}\right)$, if

$$\lambda\bar{Y}^2C_y^2+\lambda_2\bar{Y}^2C_y^2\Omega+\Re_1-1\ge 0 \quad (37)$$

Where $\Omega=\left[1-\rho_{yx}^2-\dfrac{\rho_{yz}^2\left(\rho_{yz}-\rho_{yx}\rho_{xz}\right)^2}{\rho_{yx}^2+\rho_{yz}^2-2\rho_{yx}\rho_{yz}\rho_{xz}}\right]$

Cond. (iv)

By Eq. (29) and Eq. (8), $MSE\left(t_{pro1}\right)\le MSE\left(t_M\right)$, if

$$\lambda C_y^2+\lambda_2\left\{\rho_{xz}^2C_z^2-\left(\rho_{yx}C_y-\rho_{xz}C_z\right)^2+\left(C_z-\rho_{yz}C_y\right)^2-C_y^2\rho_{yz}^2\right\}+\Re_1-1\ge 0 \quad (38)$$

Cond. (v)

By Eq. (29) and Eq. (10), $MSE\left(t_{pro1}\right)\le MSE\left(t_{Muk}\right)$, if

$$C_y^2\left[\lambda-\lambda_2\left\{\rho_{yx}^2+\rho_{yz}^2-2\rho_{yx}\rho_{yz}\rho_{xz}\right\}\right]+\Re_1-1\ge 0 \quad (39)$$

Cond. (vi)

By Eq. (29) and Eq. (12), $MSE\left(t_{pro1}\right)\le MSE\left(t_{Muk}\right)$, if

$$C_y^2\left[\lambda-\lambda_2\left\{\rho_{yx}^2+\left(\rho_{yz}-\rho_{xy}\rho_{xz}\right)^2\right\}\right]+\Re_1-1\ge 0 \quad (40)$$

Cond. (vii)

By Eq. (29) and Eq. (14), $MSE\left(t_{pro1}\right)\le MSE\left(t_{kh_1}\right)$, if

$$C_y^2\left[\lambda-\lambda_2\rho_{yx}^2-\lambda_1\rho_{yz}^2\right]+\Re_1-1\ge 0 \quad (41)$$

**b. Conditions for the Second Proposed Estimator**

Cond. (i)

Our proposed $t_{pro2}$ estimator will be efficient subject to the following conditions

By Eq. (44) and Eq. (2), $MSE\left(t_{pro2}\right) \leq MSE\left(t_r\right)$, if

$$\lambda C_y^2 + \lambda_2 C_x^2 (1-2\psi) + \frac{\Re_3}{\Re_4} - 1 \geq 0 \tag{42}$$

Here $\Re_3 = A_p D_p^2 + B_p C_p^2 + 2B_p C_p + B_p + D_p^2 - 2C_p D_p E_p - 2D_p E_p$ and $\Re_4 = A_p B_p - E_p^2 + B_p$

Condi. (ii)

By Eq. (44) and Eq. (4), $MSE\left(t_{pro2}\right) \leq MSE\left(t_p\right)$, if

$$\lambda C_y^2 + \lambda_2 C_x^2 (1+2\psi) + \frac{\Re_3}{\Re_4} - 1 \geq 0 \tag{43}$$

Condi. (iii)

By Eq. (44) and Eq. (6), $MSE\left(t_{pro2}\right) \leq MSE\left(t_{Raj}\right)$, if

$$\lambda \bar{Y}^2 C_y^2 + \lambda_2 \bar{Y}^2 C_y^2 \Omega + \frac{\Re_3}{\Re_4} - 1 \geq 0 \tag{44}$$

Where $\Omega = \left[1 - \rho_{yx}^2 - \frac{\rho_{yz}^2 \left(\rho_{yz} - \rho_{yx}\rho_{xz}\right)^2}{\rho_{yx}^2 + \rho_{yz}^2 - 2\rho_{yx}\rho_{yz}\rho_{xz}}\right]$

Condi. (iv)

By Eq. (44) and Eq. (8), $MSE\left(t_{pro2}\right) \leq MSE\left(t_M\right)$, if

$$\lambda C_y^2 + \lambda_2 \left\{\rho_{xz}^2 C_z^2 - \left(\rho_{yx} C_y - \rho_{xz} C_z\right)^2 + \left(C_z - \rho_{yz} C_y\right)^2 - C_y^2 \rho_{yz}^2\right\} + \frac{\Re_3}{\Re_4} - 1 \geq 0 \tag{45}$$

Condi. (v)

By Eq. (44) and Eq. (10), $MSE\left(t_{pro2}\right) \leq MSE\left(t_{Muk}\right)$, if

$$C_y^2 \left[\lambda - \lambda_2 \left\{\rho_{yx}^2 + \rho_{yz}^2 - 2\rho_{yx}\rho_{yz}\rho_{xz}\right\}\right] + \frac{\Re_3}{\Re_4} - 1 \geq 0 \tag{46}$$

Condi. (vi)

By Eq. (44) and Eq. (12), $MSE\left(t_{pro2}\right) \leq MSE\left(t_{Muk}\right)$, if

$$C_y^2 \left[\lambda - \lambda_2 \left\{\rho_{yx}^2 + \left(\rho_{yz} - \rho_{xy}\rho_{xz}\right)^2\right\}\right] + \frac{\Re_3}{\Re_4} - 1 \geq 0 \tag{47}$$

Condi. (vii)

By Eq. (44) and Eq. (14), $MSE\left(t_{pro2}\right) \leq MSE\left(t_{kh_1}\right)$, if

$$C_y^2 \left[\lambda - \lambda_2 \rho_{yx}^2 - \lambda_1 \rho_{yz}^2\right] + \frac{\Re_3}{\Re_4} - 1 \geq 0 \tag{48}$$

**Numerical Analysis**

To assess the performance of the proposed estimators a comparison is to be done with some of the selected prior estimators the below real data sets were utilized.

**Table 2:** description of the real data sets and their data statistics

| $y$ | $x$ | $z$ | Descriptive statistics |
|---|---|---|---|
| Data set 1: Tomato production (in tones) for Pakistan. [Source: MFA (2004)] | | | |
| For 2003. | For 2002 | For 2001 | $N = 97$, $n_1 = 30$, $n = 8$, $\bar{Y} = 3135.6186$, $\bar{X} = 3050.2784$, $\bar{Z} = 2743.9587$, $R_{yx} = 0.8072$, $R_{yz} = 0.8501$, $R_{xz} = 0.6122$, $C_y = 2.1994$, $C_x = 2.3412$, $C_z = 2.4984$. |
| Data 2:Area under wheat crop [Source: Singh and Chaudhary (1986), page 177] | | | |
| In 1974. | In 1971. | In 1973. | $N = 34$, $n_1 = 20$, $n = 7$, $\bar{Y} = 856.41$, $\bar{X} = 208.88$, $\bar{Z} = 199.44$, $C_y = 0.86$, $C_x = 0.72$, $C_z = 0.75$, $R_{yx} = 0.45$, $R_{yz} = 0.45$, $R_{xz} = 0.98$. |
| Data 3: [Source: Steel and Torrie (1960)] | | | |
| Log of leaf burn in second | Potassium percentage | Chlorine percentage | $N = 30$, $n_1 = 14$, $n = 10$, $Y = 0.6860$, $X = 4.6537$, $Z = 0.8077$, $C_y = 0.4803$, $C_x = 0.2295$, $C_z = 0.7493$, $R_{yx} = 0.1794$, $R_{yz} = -0.4996$, $R_{xz} = 0.4074$, |
| Data 4: [Source: Cochran (1977)] | | | |
| Number of "placebo" children | Number of paralytic polio cases in the "not in occulted" group | Number of paralytic polio cases in the "placebo" group | $N = 34$, $n_1 = 21$, $n=4$, $Y = 4.92$, $X = 2.59$, $Z = 2.91$, $C_y = 1.01232$, $C_x = 1.23187$, $C_z = 1.05351$, $b_{1y}=0.348$, $b_{1x}=0.272$, $b_{1z}=0.1599$, $R_{yx} = 0.7326$, $R_{yz} = 0.643$, $R_{xz} = 0.6837$. |

**Table 3:** MSE's values of all the estimators

| Estimator | Pop I | Pop II | Pop III | Pop IV |
|---|---|---|---|---|
| $t_0$ | 5818.343 | 61538.54 | 0.007237393 | 5.472026 |
| $t_R$ | 1343.636 | 58890.62 | 0.007413801 | 3.954984 |
| $t_P$ | 23769.44 | 134797.7 | 0.008477351 | 21.85731 |
| $t_{\mathrm{R}aj}$ | 6462.525 | 101607 | 0.00952368 | 7.660948 |
| $t_M$ | 4941.152 | 76390.19 | 0.02255281 | 5.577309 |
| $t_{Muk}$ | 11524.87 | 21146.49 | 0.006589888 | 2.531853 |
| $t_{Han}$ | 1065.635 | 51334.44 | 0.006120284 | 2.676178 |
| $t_{kh_1}$ | 994.2382 | 49076.98 | 0.006105306 | 2.590843 |
| $t_{kh_2}$ | 55890.58 | 531634 | 0.1062531 | 23.45319 |
| $\left(t_{\mathrm{Pro1}}\right)_{1,0}^{1/2}$ | 138.45 | 6560.041 | 0.001656238 | 0.02414931 |
| $\left(t_{\mathrm{Pro1}}\right)_{1,C_z}^{1/2}$ | 138.1026 | 6510.129 | 0.000450179 | 0.1992836 |
| $\left(t_{\mathrm{Pro1}}\right)_{\rho_{yz},C_z}^{1/2}$ | 138.068 | 6449.84 | 0.002244243 | 0.2530858 |
| $\left(t_{\mathrm{Pro2}}\right)_{1,0}^{1/2}$ | 899.422 | 17051.05 | 0.003855845 | 1.748478 |
| $\left(t_{\mathrm{Pro2}}\right)_{1,C_z}^{1/2}$ | 899.4859 | 17057.28 | 0.002737093 | 1.914389 |
| $\left(t_{\mathrm{Pro2}}\right)_{\rho_{yz},C_z}^{1/2}$ | 899.4923 | 17065.12 | 0.00536514 | 1.982269 |

**Table 4:** Percentage Relative Efficiencies relative to $t_0$

| Estimator | Pop I | Pop II | Pop III | Pop IV |
|---|---|---|---|---|
| $t_0$ | 100 | 100 | 100 | 100 |
| $t_R$ | 433.0296 | 104.4963 | 97.62054 | 138.3577 |
| $t_P$ | 24.47825 | 45.65252 | 85.37328 | 25.03522 |
| $t_{Raj}$ | 90.03204 | 60.56528 | 75.99366 | 71.42754 |
| $t_M$ | 117.7528 | 80.55817 | 32.09087 | 98.11231 |
| $t_{Muk}$ | 50.48509 | 291.0107 | 109.8257 | 216.1274 |
| $t_{Han}$ | 545.9978 | 119.8777 | 118.2526 | 204.4717 |
| $t_{kh_1}$ | 585.2062 | 125.3918 | 118.5427 | 211.2064 |
| $\left(t_{Pro1}\right)_{1,0}^{1/2}$ | 4202.487 | 938.0816 | 436.9777 | 22659.14 |
| $\left(t_{Pro1}\right)_{1,C_z}^{1/2}$ | 4213.059 | 945.2737 | 1607.67 | 2745.848 |
| $\left(t_{Pro1}\right)_{\rho_{yz},C_z}^{1/2}$ | 4214.114 | 954.1096 | 322.4871 | 2162.123 |
| $\left(t_{Pro2}\right)_{1,0}^{1/2}$ | 646.898 | 360.9076 | 187.6992 | 312.9594 |
| $\left(t_{Pro2}\right)_{1,C_z}^{1/2}$ | 646.8521 | 360.7759 | 264.4189 | 285.8366 |
| $\left(t_{Pro2}\right)_{\rho_{yz},C_z}^{1/2}$ | 646.8475 | 360.6101 | 134.8966 | 276.0487 |

**Simulations study**

Simulation studies can be a valuable tool in evaluating the performance of estimators of finite population mean in two phase sampling with two auxiliaries. In such a study, researchers could first specify the population characteristics, such as the population size, the distribution of the study variable, and the distribution of the two auxiliary variables.

Suppose a population of size N=1000 have been drawn from tri-variate normal distribution with mean vector and variance covariance matrix given as

$\mu = [8 \quad 5 \quad 4]$ and $\sum = \begin{bmatrix} 7 & 3 & 2 \\ 3 & 5 & 4 \\ 2 & 4 & 4 \end{bmatrix}$

Step-1: draw an initial sample of size $n_1$ from the above artificial population and note down the values for $x$ and $z$ and calculate their sample means.

Step-2: draw a secondary sample of size $n$ from the above $n_1$ units ($n<n_1$) and note down the values for all variables ($y$, $x$ and $z$) and calculate their second phase sample means.

Step-3: now calculate the values for the recommended and existing estimators after step 2.

Step-4: repeat step 1 to step 3 for about 10000 times.

Step-5: calculate MSE values for the all the estimators under study.

Step-6: calculate PRE values for the estimators using the below formulae

$$PRE(t_i) = \frac{\text{var}(t_0)}{MSE(t_i)} \times 100$$

Where $t_i$ is replaced by the estimator for which the PRE is to be found.

**Table 5:** MSEs of the estimators for simulated data

| Estimator | n=20 | n=50 | n=100 | n=200 |
|---|---|---|---|---|
| $t_0$ | 0.364649 | 0.16628 | 0.092379 | 0.040698 |
| $t_R$ | 0.279175 | 0.12001 | 0.057487 | 0.025756 |
| $t_P$ | 1.990844 | 1.00136 | 0.580669 | 0.264977 |
| $t_{\mathrm{R}\,aj}$ | 0.222374 | 0.09221 | 0.045019 | 0.019322 |
| $t_M$ | 0.339068 | 0.15371 | 0.084862 | 0.036589 |
| $t_{Muk}$ | 0.354949 | 0.16339 | 0.084119 | 0.038790 |
| $t_{Han}$ | 0.3215898 | 0.14568 | 0.073899 | 0.033542 |
| $t_{kh_1}$ | 0.3681529 | 0.16602 | 0.085615 | 0.039616 |
| $\left(t_{\mathrm{Pro1}}\right)_{1,0}^{1/2}$ | 0.030293 | 0.01113 | 0.00599 | 0.00265 |
| $\left(t_{\mathrm{Pro1}}\right)_{1,C_z}^{1/2}$ | 0.030092 | 0.01106 | 0.005897 | 0.00261 |
| $\left(t_{\mathrm{Pro1}}\right)_{\rho_{yz},C_z}^{1/2}$ | 0.030011 | 0.01075 | 0.00587 | 0.00259 |
| $\left(t_{\mathrm{Pro2}}\right)_{1,0}^{1/2}$ | 0.021281 | 0.00891 | 0.004938 | 0.00218 |
| $\left(t_{\mathrm{Pro2}}\right)_{1,C_z}^{1/2}$ | 0.021024 | 0.00877 | 0.00482 | 0.00213 |
| $\left(t_{\mathrm{Pro2}}\right)_{\rho_{yz},C_z}^{1/2}$ | 0.0210031 | 0.00864 | 0.00476 | 0.002105 |

**Table 6:** PREs of the estimators for simulated data

| Estimator | n=20 | n=50 | n=100 | n=200 |
|---|---|---|---|---|
| $t_0$ | 100 | 100 | 100 | 100 |
| $t_R$ | 130.6166 | 138.5526 | 162.0455 | 158.0136 |
| $t_P$ | 18.31631 | 16.60552 | 15.92821 | 15.35904 |
| $t_{\mathrm{R}\,aj}$ | 163.9801 | 180.3349 | 207.4859 | 210.6346 |
| $t_M$ | 107.5446 | 108.1432 | 109.4494 | 111.231 |
| $t_{Muk}$ | 102.7328 | 101.7686 | 110.7327 | 104.9186 |
| $t_{Han}$ | 113.3895 | 114.1407 | 125.89 | 121.3352 |
| $t_{kh_1}$ | 103.04832 | 107.375 | 108.2763 | 102.7323 |
| $\left(t_{\mathrm{Pro1}}\right)_{1,0}^{1/2}$ | 1203.728 | 1493.281 | 1540.922 | 1533.208 |
| $\left(t_{\mathrm{Pro1}}\right)_{1,C_z}^{1/2}$ | 1243.365 | 1503.208 | 1566.508 | 1559.004 |
| $\left(t_{\mathrm{Pro1}}\right)_{\rho_{yz},C_z}^{1/2}$ | 1256.628 | 1546.079 | 1574.022 | 1570.232 |
| $\left(t_{\mathrm{Pro2}}\right)_{1,0}^{1/2}$ | 1713.431 | 1866.763 | 1870.854 | 1864.067 |
| $\left(t_{\mathrm{Pro2}}\right)_{1,C_z}^{1/2}$ | 1729.434 | 1896.054 | 1915.541 | 1911.243 |
| $\left(t_{\mathrm{Pro2}}\right)_{\rho_{yz},C_z}^{1/2}$ | 1747.623 | 1923.622 | 1940.034 | 1933.431 |

## DISCUSSION AND CONCLUSION

The table 3 above displays the MSE values for estimators of the finite population mean in two phase sampling with dual auxiliaries. We have put forward two families of estimators for the finite population mean, from which several estimators can be derived. However, we have only presented three special values for each family of proposed estimators, namely one with no transformation (1,0), another with only (1,Cz), and a third with both (Ryz, Cz).

Upon comparing the MSEs in the table, it is evident that all the proposed estimators have significantly smaller MSEs compared to the estimators chosen from the literature. Moreover, the third estimator in the special cases is clearly superior to the first two estimators in both families, for all populations considered.

The table 4 illustrates the PRE values for the estimators of the finite population mean. It is important to note that the PREs are calculated relative to the usual classical estimator of the mean. It is clearly evident that all of the proposed estimators have considerably higher PREs when compared to the estimators selected from the literature. Furthermore, the proposed third estimator is particularly outstanding, as it outperforms the first two estimators in both families for all populations considered.

The table 5 provides a comprehensive overview of MSE values for various estimators, both existing and proposed, for the finite population mean. These results are based on carefully conducted simulations. The estimators were rigorously evaluated using a first phase sample of varying sizes, 20, 50, 100, and 200, with a second phase sample taken as half of each first phase sample. It is evident that, across different sample sizes, all the six special cases of the proposed families of estimators outperform all other mentioned estimators in terms of MSEs for the same parameters.

The table 6 offers a thorough evaluation of the PRE values for a range of estimators, both existing and proposed, for the finite population mean. The results are based on meticulously conducted simulations that accurately reflect real-world scenarios.

Upon scrutinizing the table, it becomes evident that based on artificial data all six special cases of the suggested classes of estimators outperform all other estimators mentioned in terms of PREs for the same parameters. This undeniably establishes the proposed families of estimators as an exceptionally efficient and consistent choice for estimating the finite population under two-phase sampling with two auxiliaries.

Overall, these results provide strong evidence in support of the effectiveness and superiority of our proposed estimators and we can confidently conclude that our proposed estimators are not only highly efficient but also superior to the previously established estimators for estimating the finite population mean in two phase sampling with dual auxiliaries.

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