Kurdish Studies
May 2024
Volume: 12, No: 4, pp. 523-534
ISSN: 2051-4883 (Print) | ISSN 2051-4891 (Online)
www.KurdishStudies.net

DOI: 10.53555/ks.v12i4.2994

Parameter Estimation for Exponential Distributions via Moment Generating Functions in the Presence of Outliers

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ABSTRACT
For data mining, outlier identification depends on distance measurements, grouping, and statistical approaches. The outliers change the behavior of the data and cause a serious error in the estimated results, that is why the parameters will be estimated after eliminating the effect. Outlying values have been an issue of concern to researchers and data analysts. The outliers will be detected and find out the effect of outliers and define how outliers affect the data set. Some methods will be used like MME and MLE to estimate the parameters of different distributions having outlier effects. The behavior of the parameters will be discussed in this research as then eliminate the outlier's effect on the data set. The study of multiple outliers is imperative although outlying values can contribute to misrepresentation of the formula, wrong parameterization, and inaccurate summary statistics. A simulated study was carried out to investigate the strength of the test statistics. Exponential distribution discussed and calculated their parameters in size-biased as well as area-biased functions. The outliers have been discussed in this research by using discordancy tests. The graphical representation is used to differentiate the behavior of scale and shape parameters. The outliers have been detected with the help of the discordancy test for the single outlier as well as for two outliers and the results.

Keywords: Generalized Exponential Distribution, Probability Density Function, Detect Outliers, Area & Size Biased Exponential Distribution

Introduction

In any data, an outlier is a value or point that significantly differs from the rest of the given values. These outliers may or may not result from variability in data collection or calculation; they can also stem from experimental or instrumental errors. Sometimes, these numbers or letters are either included or excluded from the dataset. A single outlier can be harmful and cause significant problems in statistical analysis, presenting a fundamental challenge in various financial research endeavors. Detecting outliers is an essential initial step in many data mining applications. Nowadays, various methods exist for identifying or removing outliers, distinguishing between univariate and multivariate techniques, as well as non-parametric and parametric methods. In numerous data analysis scenarios, a vast number of events are collected, which may be sampled. The first coherent step in performing analysis is detecting observations with outliers. While outliers are often considered noise or errors, sometimes these values carry important information.

Outliers represent a prominent topic of discussion within the field of statistics. However, it's essential to acknowledge that not every dataset or statistical analysis will inevitably contain one or more outlier observations. Typically perceived as disruptions within the data, outliers often provoke curiosity about their potential significance. This research endeavors to demonstrate that outliers can offer valuable insights when scrutinized deliberately post-identification through diligent detection efforts. Parameters may deviate as a consequence of outliers, thereby influencing the accuracy of the resulting analysis. Consequently, it is imperative to assess and quantify the impact of these observations.

An outlying observation, often referred to as an outlier, is characterized by its noticeable deviation from the rest of the data points within the sample it belongs to, as highlighted by (Barnett and Lewis, 1994).

In the presence of outliers, the validity of propositions may vary. Grubbs (1969) investigated outliers within observations where proportions varied. Hawkins (1980) employed various methods to detect outliers in datasets and emphasized comparing them with real-life applications. Miller (1981) found that observations often do not align when outliers are present.

Currently, the sample mean, and variance provide us with precise estimates of the data's central tendency and spread, assuming outliers are not removed. An outlier suggests a deviation that may indicate inaccuracy or inconsistency compared to the rest of the data points. While numerous definitions of outliers exist, one common interpretation is: "An observation, or a subset of observations, that seems inconsistent with the majority of the dataset "(Barnett and Lewis (1994).

An outlier, as defined by (Aggarwal and Yu, 2011), is a data point that significantly deviates from the rest of the dataset according to a certain measure.

Identified outliers represent potential instances of aberrant data, capable of detrimentally influencing model accuracy, skewing parameter estimation, and yielding erroneous outcomes. Consequently, their early detection prior to modeling and analysis is imperative (Williams *et al.,* 2010).

Methods for Identification of Outlier in past

The outlier was a major problem in literature identified in parameter estimation. Mahala Nobis provided that the solution of suggested problem firstly by using the point of 0 in dataset. Campbell extended this work and determined the estimators in the presence of outliers. These estimators namely called M-estimators, for f(x) and g(x). For the identification of the outliers two methods are presented in literature.

The following stages are given below.

- Phase I: Considered the estimation of a location and scale.
- Phase II: Scale and location estimation suggested some points for identification of outliers.

Moment Distribution

In observational studies concerning human life, wildlife, plants, angles, and creepy crawlies, there lacks a well-defined sampling framework. In such populations, selecting units of equal probability isn't feasible. It's impossible to compile a collection encompassing all modules within the population due to biased historical observations. If biases go unnoticed, serious complications arise, leading to incorrect conclusions. In ecological research, observations fall into non-experimental, unsystematic, and non-simulated categories. Issues of model selection and data interpretation receive considerable attention to address these challenges. Researchers have proposed several ad hoc solutions to correct bias, including weighted distributions. Weighted distributions arise when historical observations are non-randomly generated.

Researchers and analysts have defined a variety of ad hoc arrangements to rectify inclinations, leading to the concept of weighted conveyance. Minute discrepancies arise when observations are not generated randomly. Let's Assume that although the probability density function (pdf) of the random variable x is $f(x;\theta)$ with unknown parameterθ. The given below distribution, called moment distribution,

$$g(x;\theta) = \frac{w(x)f(x;\theta)}{E[w(x)]}$$

Where:

w(x): non-negative weight function.

The weight function defined as $w(x) = x^s$

Then the resulted distributions (size-biased distributions of order s) are defined as:

$g(x;\theta) = \frac{x^s f(x;\theta)}{\mu_s'}$; Where $\mu_s' = \int x^s f(x;\theta)$

Weighted Distribution

Mathematically, the weighted distribution with probability function as:

$$f(x) = \frac{w(x)f_0(x)}{E[w(x)]} \quad x > 0$$

Where, $w(x)$ is the weight function and $f_0(x)$ is the probability density function.

Size Biased Distribution

The concept of one-sided dissemination, as articulated by Fisher in 1934, presents a unique form of weighted dissemination. This method is employed to ascertain bias within a given dataset. Biased distributions occur when observations derived from a non-random method are not equally likely to be recorded and are instead recorded according to a specific weighting function. This is particularly pertinent in fields such as forestry, medical sciences, and psychology. Length one-sided testing occurs when the probability of selecting an individual in a population is

proportional to its magnitude. Conversely, length-biased distributions arise when observations are selected with probability proportional to their magnitude.
Mathematically the size biased distribution with probability function as:

$$f^*(x;\delta) = \frac{x.f(x,\delta)}{E(x)}$$

Where $E(x) = \int x.f(x,\delta)dx$
An observer documents a concept through nature, aligning with a specific deterministic demonstration, indicating that the documented perception lacks initial distribution until a split is granted to each perception, even if it might be reported. Over the past decade, the majority of publications have consistently employed principles of weighted and size-biased research methodologies. In light of this, I examine several common models leading to weighted distributions, typically constrained by unity. These models encompass likelihood testing, additive damage models, visibility biases contingent on data essence, compilation, and two-stage inspection. They document several significant publications and their size-biased forms. Discrepancies between the harsh values of two weighted distributions are frequently postulated. These findings serve to assess population data and wildlife management (Patil and Rao, 1978).

Area Biased Distribution

Area-biased distributions categorize sampled data with probability proportional to their lengths, leading to inherent bias. Introduced by Cox in 1962, they find applications in various fields like biomedical research, family background analysis, and predictive modeling. Further developments include weighted(moment) distributions, notably area biased distributions, pioneered by Patill and Rao. These encompass well-known distributions like beta, gamma, and Pareto. Recent research extends this concept to lifetime distributions, including length-biased weighted generalized Rayleigh, Bayes estimation of length-one-sided Weibull, and length-biased beta distributions.
Similarly, area biased distribution assumed that the square of random variable x of size biased distribution. Assume the irregular variable (x) contains a likelihood distribution work $f(x;\theta)$ with undetermined parameter. The given below distribution, called moment distribution (area biased distribution).

$$g(x;\alpha,\beta) = \frac{w(x)f(x;\alpha,\beta)}{E[w(x)]}$$

$g(x;\theta) = \frac{x^s f(x;\theta)}{\mu_s'}$;
Where $\mu_s' = E(x^s) = \int_{-\infty}^{\infty} x^s f(x;\theta)\,dx$ is the s^{th} raw moment of $f(x;\theta)$.
When s = 1then $g(x;\theta)$ turned into length- biased or size-biased distribution and when s = 2 then $g(x;\theta)$ is named as area-biased distribution. The expression for area biased given as:

$$f^*(x;\delta) = \frac{x^2.f(x,\delta)}{E(x^2)}$$

Taillie et al. (1995) showcased how fish stocks in populations, utilizing weighted distributions in fisheries datasets, weren't directly linked to test selection strategy. Instead, they served as beneficial models for observed data. Recognizing their utility, Gove and Patil (1998) devised a coherent hypothesis unifying DBH-frequency and basal area-DBH distributions, leveraging the quadratic relationship between diameter and basal area.

How to Identify the Outliers

There are some methods or test statistic to identify the outliers. Some of these tests statistic are given below:

i. The Dixon type test
ii. Likelihood of maximum ratio (MLR) statistic test
iii. Zerbet as well as Nikulin (ZN) test statistic
iv. Shadrokh and Pazira test statistic
v. Lalitha and Kumar test statistic
vi. Gap test statistic
vii. Tietjen-Moore test statistic

Outlier Detecting Methods

Exception labeling strategies are employed for two primary reasons. Firstly, they distinguish outliers before applying formal statistical tests. Additionally, they identify extreme values far from the dataset's distribution. Although common, after formal testing, outliers are only observations deviating from expected distributions. Moreover, when large datasets have ambiguous distributions, labeling strategies are more beneficial.

Informal Methods
In this segment we donate brief rundown of casual strategies of outlier discovery. The foremost common and valuable strategies are recorded here. The advantage of these strategies are they valuable and simple to get it.

Standard Deviation (SD) Method
The SD method is used for detecting the outliers from a given dataset. The expressions are given below:
2 Standard Deviation (SD) Method:

$$\bar{x} \pm 2s$$

3 Standard Deviation (SD) Method:

$$\bar{x} \pm 3s$$

Where $\bar{x}$ and s are the test cruel test standard deviation (SD). In this approach outside observations from this interval considered as outliers. The Chebyshev's inequality defined as the off chance that an arbitrary variable x with cruel μ and fluctuation σ^2 exists, then for any$k > 0$,

$$P[|x-\mu| \geq k\sigma] \leq \frac{1}{k^2}$$

$P[|x-\mu| < k\sigma] \geq 1-\frac{1}{k^2}$ $\qquad k > 0$

Z-Score
For screening the data from outliers, the method named as Z-score method under mean and standard deviation. $Z_i = \frac{x_i-\bar{\bar{x}}}{s}$, where $x_i \sim N(\mu, \sigma^2)$. The main objective of this approach is defined as $x_i \sim N(\mu, \sigma^2)$, where the $Z_i$approximated to standard normal distribution. As z-score value increases from 3, it examined that the dataset contained outlier.

Modified Z-Score
In Z-score strategy two estimators test mean and test standard deviation are utilized and both estimators can be influenced by several extraordinary values or by indeed a single extraordinary value. To maintain a strategic distance from this issue, the middle and the middle of the variance of absolute of the middle (MAD) are utilized within the altered Z-Score rather than the mean as well ass standard deviation (SD) of the test, separately (Iglewicz and Hoaglin, 1993).
MAD = Median$|x_i - \tilde{x}|$
Where
$\tilde{x}$: sample median
The customized Z-Score is given as:

$$M_i = \frac{0.6745(x_i - \bar{x})}{MAD}$$

Where *E(MAD)=0.6745σ* for large extreme normal data.

Formal Method
Informal outlier detection methods typically yield intervals due to their lack of reliance on test statistics, rendering them unreliable. Hence, formal methods, reliant on test statistics and distributional assumptions, are preferred. These methods, commonly used for hypothesis testing, vary in their applicability to different sample sizes.
Mekay (1935) proposed two test statistics $B_1 = \frac{x_n-\bar{x}}{\sigma}$ and $B_2 = \frac{\overline{x_n}-x_1}{\sigma}$ to test smallest and largest observations as outliers.
Irwin (1925) proposed two statistics $B_1 = \frac{x_n-x_{n-1}}{\sigma}$ and $B_2 = \frac{x_2-x_1}{\sigma}$ to test smallest and largest observations as outliers.

Likelihood Function
Many probability distributions involve unclear parameters. We estimate these parameters using sample data. The Likelihood function indicates how well these parameters are captured by the results. Specifically, it identifies the most probable parameter values based on observed data, making it a crucial tool in data science. The Fisher information matrix, a key metric in maximum likelihood estimation, measures the independence between estimated parameters. Its diagonalization ensures parameter estimates are independent but doing so "locally" or at specific parameter values is illogical as those values are unknown and must be estimated.

Suppose $(X_1, X_2, \dots, X_n) \sim F_\theta$, where θ is unknown. With the time being, we expect that θ exist in a subset of function R. And further assume that, for eachθ_i, $F_\theta(x)$ admits a PMF/PDF$F_\theta(x)$. By the assumed independence, the joint distribution of $(X_1, X_2, \dots, X_n)$ is characterized by

$$f_\theta(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} f_\theta(X_i)$$

i.e., autonomy means accumulate." In the Stat context, it is understanding the above expression to be a function of $(X_1, X_2, \dots, X_n)$ for fixed θ.
Definition.
If $(X_1, X_2, \dots, X_n) \overset{iid}{\sim} f_\theta$, then the likelihood function is.

$$L(\theta) = f_\theta(X_1, X_2, \dots, X_n) = \prod_{i=1}^{n} f_\theta(X_i)$$

Behaves as a function of θ in what takes after, it may infrequently add subscripts i.e., $L_x(\theta)$ or$L_n(\theta)$, to point out the obsession of the likelihood on data $X = (X_1, X_2, \dots, X_n)$ or on sample size *n*. Also, can be written as:

$$l(\theta) = logL(\theta)$$

Objectives

- ✓ To estimate joint distribution and the parameters when outliers in probability distributions.
- ✓ Evaluate the effect of outliers in moment distributions.

Review of Literature

Kimber (1982) a procedure which sequential for the test of k upper outlier in exponential sample is propsed, K=2,3,4 is under considration and critaical values are tabulated. Existing test are also comparied and flexibality is also discussed, low outliers are also considered in study.

Pettit (1988) study a approach of bayesian for outliers models of exponential family. This approach is useful when analyst prior knowledge in such manner that any observation will be surprising with big probability. Predictive ordinate is used for measuring the most surprising observation, must lie outside the sample. Models of bayes factores are given for exponential sample with outlier and also without outlier.

Garcia *et al.* (1990) present an outline which gives us an idea to analyses in ordinary way the error which can affect the exponential distribution. On the base of this study outlier definition is stated and also typify the identification problems of outliers. This study gives an idea to construct a framework in general which carry out qualitative data analysis which when apply to outliers analysis shows benefits with classical approach respect.

Hossain *et al.* (1997) study about the estimation of the paramter with unweighted least squares of two paramters Burr XII distribution and also compaed with the maximum liklihood and maximum products of spacings. These estimation method are tested with and also without outliers in simulation study, also calculate the confidenece interval for C and also for K. after studying colclude that maximum liklhood is littlte better than least squares.

Schultze *et al.* (2000) study the outliers identification of exponential samples conception ally in sense of David and Gather (1989, 1993) by means of so called outlier region. Empirical version of such a region also named as outlier identifier in exponential distribution, it is mainly based on unknown scale parameter. The bad behavior of different outlier identifier is compared and concludes that robust estimator for scale is best. Finding of the study is based on standardized version of sample median.

Dixit *et al.* (2000) maximum likelihood mixture and moments of the estimators are calculated for exponential distribution samples when outliers are in the data which was obtained from uniform probability distribution, further estimators are being compared when all of the parameters are not known determinants and biases are under study numerical technique is used for this. The results of the study shows that estimators are opportunistic unbiased, in the last of the study expressed that mixture estimators are good than maximum likelihood and moments estimators.

Pazira (2001) generalized exponential distribution discussed about estimated probability i.e. $P(Y < X)$ when the outliers are in data and also when scale parameter is known. He fined the MLE of the pdf of *R* and does the numerical experiments by using MATLAB and concludes that the maximum likelihood estimator *R* gives results better than other methods.

Hossain *et al.* (2011) estimate parameters when the outliers exist in the values of burr XII distribution by using MLE, least square and maximum product spacing then make comparison of the results of given methods and conclude values of estimates varied on the methods used and the RMSE's decreased as the ample size n increased.

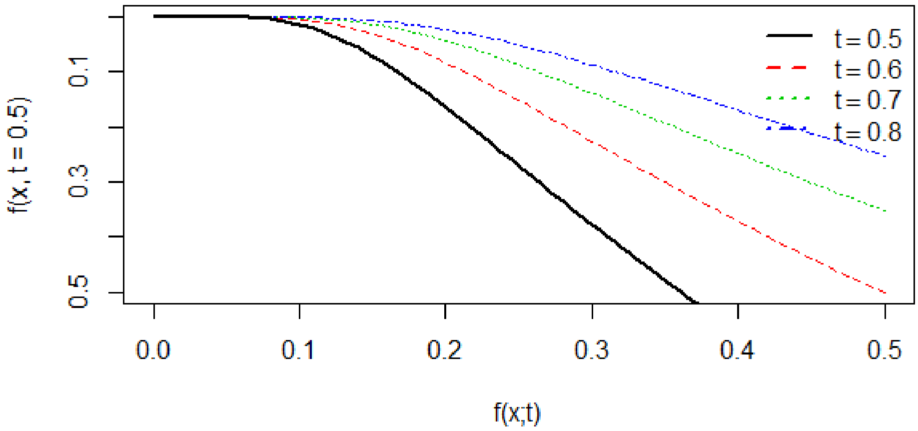
Kochar *et al.* (2011) study that hetrogeneous exponential paraell system is more skwed in the compsrion of independent and idenically exponential distributed compenents. This paper is for the propose of extion the pervious study when there are only two type of component in the system.

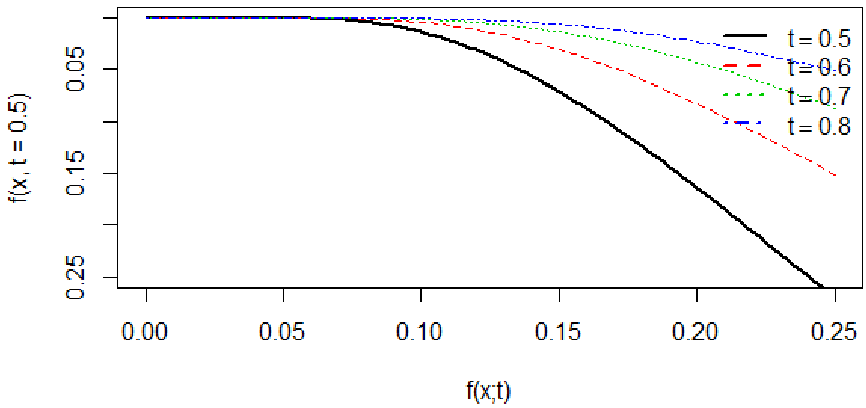
Nasiri (2018) the scale parameter estimation in exponential distribution with given prior results are obtained of the exponential distribution when outliers are exist by the estimator of Bayesian shrinkage. And this shrinkage estimator

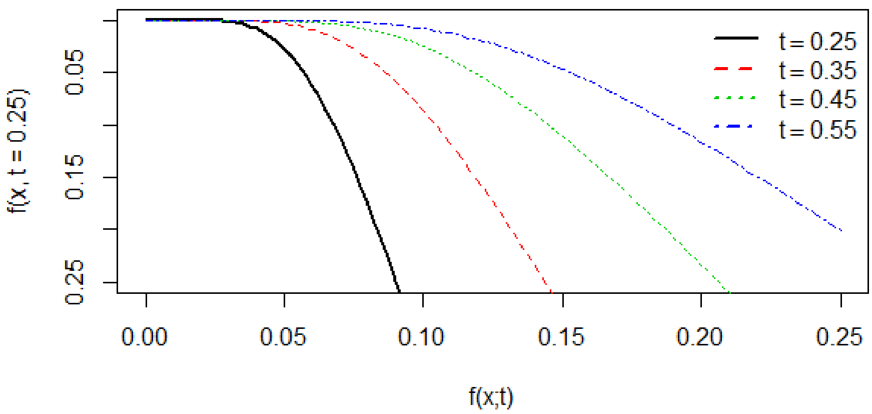
is calculated for two purposes. I.e. for the LINEX loss function and for the parameter estimation with the outlier of exponential case. The comparison is done for the appropriate estimator depends upon the LINEX loss function with other estimations methods. Numerical study is used to compare the results of these estimators.
Nasiri (2019) the interval shrinkage estimators of scale exponential distribution's parameters when their Outliers are discussed. He also discusses the joint distribution and minimizes the MSE of the estimators within the occurrence of outliers.

Exponential Distribution:

$f(y;\varphi) = \frac{1}{\varphi} e^{-\frac{x}{\varphi}}$ Where $y \geq 0$ and $\varphi > 0$







Moment Exponential Distribution:

The probability density function of Moment Exponential distribution is defined as:

$g(x;t) = \frac{x^h f(x;t)}{\mu'_h}$; $\quad x > 0, t > 0, h = 1,2,3, \ldots$

Where $\mu'_h = E(x^h) = \int_{-\infty}^{\infty} x^h . f(x;t)\, dx$

$f(x;t) = \frac{1}{t} e^{-\frac{x}{t}}$ Where $x \geq 0$ and $t > 0$

$$\mu'_h = \int_0^{\infty} x^h . \frac{1}{t} e^{-\frac{x}{t}}\, dx$$

$$\mu'_h = \frac{1}{t} \int_0^{\infty} x^h e^{-\frac{x}{t}}\, dx$$

Put $\frac{x}{t} = b \;\Rightarrow\; x = bt$

$dx = tdb$

$x \to 0$ *then* $b \to 0$ Also $x \to \infty$ *then* $b \to \infty$

$$\mu'_h = \frac{1}{t} \int_0^{\infty} (bt)^h e^{-b}\, tdb$$

$$= t^h \int_0^{\infty} (b)^h e^{-b}\, db$$

$$\mu'_h = t^h \Gamma(h+1)$$

$$g(x;t) = \frac{x^h . \frac{1}{t} e^{-\frac{x}{t}}}{t^h \Gamma(h+1)}$$

$g(x;t) = \frac{x^h e^{-\frac{x}{t}}}{t^{h+1}\Gamma(h+1)}$ .... (A)

$$\mu'_r = E(x^r) = \int_{-\infty}^{\infty} x^r . g(x)\, dx$$

$$\mu'_r = \int_0^{\infty} x^r . \frac{x^h e^{-\frac{x}{t}}}{t^{h+1}\Gamma(h+1)}\, dx$$

$$= \frac{1}{t^{h+1}\Gamma(h+1)} \int_0^{\infty} x^{r+h} e^{-\frac{x}{t}}\, dx$$

Put $\frac{x}{t} = k \;\Rightarrow\; x = tk$

$dx = tdk$

$x \to 0$ *then* $k \to 0$ Also $x \to \infty$ *then* $k \to \infty$

$$\mu'_r = \frac{1}{t^{h+1}\Gamma(h+1)} \int_0^{\infty} (tk)^{r+h} e^{-b}\, tdk$$

$$\mu_r' = \frac{t^r}{\Gamma(h+1)}\int_0^{\infty} (k)^{r+h} e^{-b}\, dk$$

$\mu_r' = \frac{t^r\Gamma(r+h+1)}{\Gamma(h+1)}$ ……. (a)

By using $r = 1,2,3,4$ in equation (a) and get:

$$\mu_1' = t(h+1)$$
$$\mu_2' = t^2(h+1)(h+2)$$
$$\mu_3' = t^3(h+1)(h+2)(h+3)$$
$$\mu_4' = t^4(h+1)(h+2)(h+3)(h+4)$$

First four moments of mean are:

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$\mu_2 = t^2(h+1)(h+2) - [\theta(h+1)]^2$$
$$\mu_2 = t^2(h+1)$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$\mu_3 = 2t^3(h+1)$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$
$$\mu_4 = 3t^4(h^2 + 4\text{h} + 3)$$

$\beta_1 = \frac{4}{h+1}$ $\gamma_1 = \frac{2}{\sqrt{h+1}}$

$\beta_2 = \frac{3(h+3)}{h+1}$ $\gamma_2 = \frac{6}{h+1}$

Size Biased Exponential Distribution:

The probability density function of size biased Exponential distribution is obtained by taking $s = 1$ in eq. (A)

$$g(x;t) = \frac{x^h e^{-\frac{x}{t}}}{t^{h+1}\Gamma(h+1)}$$

$$g(x;t) = \frac{xe^{-\frac{x}{t}}}{t^2}$$

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r.\, g(x;t)\, dx$$

$$\mu_r' = \int_0^{\infty} x^r.\frac{xe^{-\frac{x}{t}}}{t^2\Gamma 2}\quad dx$$

$$\mu_r' = \frac{t^r}{\Gamma 2}\int_0^{\infty} x^{r+1} e^{-\frac{x}{t}}\, dx$$

$\mu_r' = t^r\Gamma(\text{r}+2)$ ……. (b)

By using $r = 1,2,3,4$ in equation (b) and get:

$$\mu_1' = 2t$$
$$\mu_2' = 6t^2$$
$$\mu_3' = 24t^3$$
$$\mu_4' = 120t^4$$

For first four moments of mean are:

$$\mu_1 = 0$$
$$\mu_2 = \mu_2' - (\mu_1')^2$$
$$\mu_2 = 2t^2$$
$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu_1')^3$$
$$\mu_3 = 16t^3$$
$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3(\mu_1')^4$$
$$\mu_4 = 24t^4$$

$\beta_1 = 2$ $\gamma_1 = \sqrt{2}$

$\beta_2 = 6$ $\gamma_2 = 3$

Discordancy Test for Single Upper Outlier (k=1)

The probability density function of Size Biased Exponential Distribution is,

$g(x;t) = \frac{xe^{-\frac{x}{t}}}{t^2}$; $\quad x > 0, t > 0$

To build the test statistics for a single upper outlier test, say x_n in a biassed exponential sample size. Our hypothesis is

$H_0: x_i \epsilon G \quad i = 1,2,3, \dots, n$

Proclaimed that all of the results belong to the density distribution G

$g(x;t) = \frac{xe^{-\frac{x}{t}}}{t^2}$; $\quad t > 0$

Consider here a slippage alternative hypothesis H_1 that $(n-1)$ findings refer to G and one observation claims x_{n-1} to Size Biased Exponential G_1 with density distribution.

$g(t;\theta) = \frac{\theta^2 xe^{-\frac{\theta x}{t}}}{t^2}$; $\quad t > 0, \theta < 1$

here θ is a slippage parameter, it may write.

$H_0: \theta = 1$

$H_1: \theta < 1$

The log likelihood function under H_0 is

$$L_{H_0}(t) = \sum ln\, x_i - 2nlnt + \frac{\sum x}{t}$$

Where $\bar{x}$ is sample mean of all observations. $L_{H_0}(t)$ Is maximized at $t = \frac{\bar{x}}{2}$

So,

$$\hat{L}_{H_0}(t) = \sum ln\, x_i - 2nln2 - 2nln\bar{x} - 2n$$

The log likelihood function under H_1 is

$$L_{H_1}(t,\theta) = \prod \frac{\theta^2 xe^{-\frac{\theta x}{t}}}{t^2}$$

$$L_{H_1}(t,\theta) = 2ln\theta + \sum ln\, x_i - \frac{\theta x_n}{t} - \frac{(n-1)\bar{x}'}{t} - 2nlnt$$

Where $\bar{x}'$ is sample mean of (n-1) observations. $L_{H_1}(t,\theta)$ Is maximized at $\hat{t} = \frac{\bar{x}}{2}$ and $\hat{\theta} = \frac{x_n}{\bar{x}'}$ if $x_n > \bar{x}'$ then,

$$\hat{L}_{H_1}(t,\theta) = 2nln2 - 2nln\bar{x}' + 2lnx_n + \sum ln\, x_i - 2ln\bar{x}' + 2n$$

The LLRT is $\widehat{\Lambda} = (\hat{L}_{H_1} - \hat{L}_{H_0})$. Given ratio equal to zero if $x_n < \bar{x}'$ while if $x_n > \bar{x}'$ then.

$$\widehat{\Lambda} = 2nln\bar{x} - 2nln\bar{x}' + 2lnx_n - 2ln\bar{x}'$$

$$\widehat{\Lambda} = 2nln\bar{x} - 2(n+1)ln\bar{x}' + 2lnx_n$$

Discordancy test for tow upper outlier (k=2)

The pdf of SBED is,

$g(x;t) = \frac{xe^{-\frac{x}{t}}}{t^2}$; $\quad x > 0, t > 0$

To test a single upper outlier called x_n in a SBE sample. There is hypothesis given as

$H_0: x_i \epsilon G \quad i = 1,2,3, \dots, n$

Proclaimed that all of the results belong to the density distribution G with the density.

$g(x;t) = \frac{xe^{-\frac{x}{t}}}{t^2}$; $\quad t > 0$

Consider here a slippage alternative hypothesis H_1 that $(n-2)$ findings refer to G and one observation claims x_{n-1} to SBE G_1 with density distribution.

$g(t;\theta) = \frac{\theta^2 xe^{-\frac{\theta x}{t}}}{t^2}$; $\quad t > 0, \theta < 1$

here θ is a slippage parameter that may write.

$$H_0: \theta = 1$$
$$H_1: \theta < 1$$

The log likelihood function under H_0 is.

$$L_{H_0}(t) = \sum ln\, x_i - 2nlnt + \frac{\sum x}{t}$$

Where $\bar{x}$ is sample mean of the given values. $L_{H_0}(t)$ is maximized at $t = \frac{\bar{x}}{2}$
So,

$$\hat{L}_{H_0}(t) = \sum \ln x_i - 2n\ln 2 - 2n\ln\bar{x} - 2n$$

The log likelihood function under H_1 is.

$$L_{H_1}(t,\theta) = \prod \frac{\theta^2 x e^{-\frac{\theta x}{t}}}{t^2}$$

$$L_{H_1}(t,\theta) = 4\ln\theta + \sum \ln x_i - \frac{2\theta\bar{x}''}{t} - \frac{(n-2)\bar{x}'}{t} - 2n\ln t$$

Where $\bar{x}''$ is sample mean of (n-2) values and $\bar{x}'' = \frac{x_n + x_{n-1}}{2}$. $L_{H_1}(t,\theta)$ is maximized at $\hat{t} = \frac{\bar{x}}{2}$ and $\hat{\theta} = \frac{\bar{x}'}{\bar{x}''}$ if $x_n > \bar{x}'$ then,

$$\hat{L}_{H_1}(t,\theta) = 2n\ln 2 - 2n\ln\bar{x}' + 2\ln x_n + \sum \ln x_i - 2\ln\bar{x}' + 2n$$

The LLRT is $\hat{\Lambda} = (\hat{L}_{H_1} - \hat{L}_{H_0})$. Given ratio equal to zero if $x_n < \bar{x}'$ while if $x_n > \bar{x}'$ then

$$\hat{\Lambda} = 2n\ln\bar{x} - 2n\ln\bar{x}' + 2\ln x_n - 2\ln\bar{x}'$$

$$\hat{\Lambda} = 2n\ln\bar{x} - 2(n+1)\ln\bar{x}' + 2\ln x_n$$

Area Biased Exponential Distribution:

The pdf of ABED is obtained by taking $s = 2$ in eq. (A)

$$g(x;t) = \frac{x^2 e^{-\frac{x}{t}}}{t^3\Gamma 3}$$

$$\mu_r' = E(x^r) = \int_{-\infty}^{\infty} x^r . g(x;t)\,dx$$

$$\mu_r' = \int_0^{\infty} x^r . \frac{x^2 e^{-\frac{x}{t}}}{t^3\Gamma 3}\,dx$$

$$\mu_r' = \frac{1}{t^3\Gamma 3}\int_0^{\infty} x^{r+2} e^{-\frac{x}{t}}\,dx$$

$\mu_r' = \frac{t^r\Gamma(r+3)}{2}$ ……. (c)

By using $r = 1,2,3,4$ in equation (c) and get:

$$\mu_1' = 3t$$
$$\mu_2' = 12t^2$$
$$\mu_3' = 60t^3$$
$$\mu_4' = 360t^4$$

First moments of mean are:

$$\mu_1 = 0$$
$$\mu_2 = 12t^2 - (3t)^2$$
$$\mu_2 = 3t^2$$
$$\mu_3 = 6t^3$$
$$\mu_4 = 45t^4$$

$\beta_1 = \frac{4}{3}$ $\qquad \gamma_1 = \frac{2}{\sqrt{3}}$

$\beta_2 = 5$ $\qquad \gamma_2 = 2$

Discordancy Test for Single Upper Outlier (k=1)

To develop the test statistic in ABED. To test of a single outlier says as x_n in ABS sample. The hypothesis is.

$H_0: x_i \epsilon G \qquad i = 1,2,3,\dots,n$

Exposed that the given values belong to distribution *G* with density function.

$g(x;t) = \frac{x^2 e^{-\frac{x}{t}}}{2t^3}; \qquad x > 0, t > 0$

Suppose there is a slippage alternative hypothesis H_1 that $(n-1)$ of the values belonged to *G* and an observation called x_n to ABGD $G_1$with density.

$g(x;t) = \frac{x^2\theta^3 e^{-\frac{\theta x}{t}}}{2t^3}; \qquad x > 0, \theta < 1$

Here θ is a slippage parameter may write as
$H_0: \theta = 1$
$H_1: \theta < 1$
The log likelihood function under H_0 is

$$L_{H_0}(t) = \sum ln\, x^2 - 3nlnt - \frac{\sum x}{t} - nln2$$

Where $\bar{x}$ is sample mean of total values. $L_{H_0}(t)$ is maximized at $t = \frac{\bar{x}}{3}$
So,

$$\hat{L}_{H_0}(t) = \sum ln\, x^2 - 3n - nln2 - 3nln\bar{x} + 3nln3$$

The log likelihood function under H_1 is

$$L_{H_1}(t) = \prod \frac{x^2\theta^3 e^{-\frac{\theta x}{t}}}{2t^3}$$

$$L_{H_1}(t) = 2nlnt + 2ln\theta + \sum ln\, x_i - t(n-1)\bar{x}' - \theta t x_n$$

Where $\bar{x}'$ is mean of sample values of (n-1) observations. $L_{H_1}(t)$ is maximized at $t = \frac{2}{\bar{x}'}$ and $\hat{\theta} = \frac{x_n}{\bar{x}'}$ if $x_n > \bar{x}'$ then,

$$\hat{L}_{H_1}(t) = 2nln2 - 2nln\bar{x}' + 2lnx_n + \sum ln\, x_i - 2ln\bar{x}' + 2n$$

The LLRT is $\hat{\Lambda} = (\hat{L}_{H_1} - \hat{L}_{H_0})$. This ratio equal to zero if $x_n < \bar{x}'$ while if $x_n > \bar{x}'$ then

$$\hat{\Lambda} = 2nln\bar{x} - 2nln\bar{x}' + 2lnx_n - 2ln\bar{x}'$$

$$\hat{\Lambda} = 2nln\bar{x} - 2(n+1)ln\bar{x}' + 2lnx_n$$

Discordancy test for two upper outliers (k=2)

To develop the test statistic in ABED. To test of upper two outliers says as x_n in area biased Exponential sample. Our hypothesis is
$H_0: x_i \epsilon G \qquad i = 1,2,3, \dots, n$
Exposed that the total values belong to distribution *G* with density

$g(x;t) = \frac{x^2 e^{-\frac{x}{t}}}{2t^3}; \qquad x > 0, t > 0$

Suppose a slippage alternative hypothesis here H_1 that $(n-2)$ of the values belonging to *G* and only single observation called as x_{n-1} to ABGD $G_1$with density function

$g(x;t) = \frac{x^2\theta^3 e^{-\frac{\theta x}{t}}}{2t^3}; \qquad x > 0, \theta < 1$

Here θ is a slippage parameter may write as
$H_0: \theta = 1$
$H_1: \theta < 1$
The log likelihood function under H_0 is

$$L_{H_0}(t) = \sum ln\, x^2 - 3nlnt - \frac{\sum x}{t} - nln2$$

Where $\bar{x}$ is sample mean of total values. $L_{H_0}(t)$ is maximized at $t = \frac{\bar{x}}{3}$
So,

$$\hat{L}_{H_0}(t) = \sum ln\, x^2 - 3n - nln2 - 3nln\bar{x} + 3nln3$$

The log likelihood function under H_1 is

$$L_{H_1}(t) = \prod \frac{x^2\theta^3 e^{-\frac{\theta x}{t}}}{2t^3}$$

$$L_{H_1}(t) = 2nln\beta + 2ln\theta + \sum ln\, x_i - t(n-1)\bar{x}' - \theta t x_n$$

Where $\bar{x}'$ is sample mean of (n-1) values. $L_{H_1}(\beta)$ is maximized at$t = \frac{2}{\bar{x}'}$ and $\hat{\theta} = \frac{x_n}{\bar{x}'}$ if $x_n > \bar{x}'$ then,

$$\hat{L}_{H_1}(t) = 2nln2 - 2nln\bar{x}' + 2lnx_n + \sum ln\, x_i - 2ln\bar{x}' + 2n$$

The LLRT is $\hat{\Lambda} = (\hat{L}_{H_1} - \hat{L}_{H_0})$. This ratio equals to zero if $x_n < \bar{x}'$ while if $x_n > \bar{x}'$ then

$$\hat{\Lambda} = 2nln\bar{x} - 2nln\bar{x}' + 2lnx_n - 2ln\bar{x}'$$

$$\widehat{\Lambda} = 2nln\bar{x} - 2(n+1)ln\bar{x}' + 2lnx_n$$

Conclusions

This research leads to analyst how much impact may well be disposed of by the simulation and discordancy tests. The moment distributions are more fitting for the analysts of forestry as well as for the engineering students. Exponential distribution is the foremost commonly utilized within the field of forestry and broadly utilized within research. Sometime a mistake happens within the information which is due to the instrumental mistake which cause genuine alter within the information and it may alter the shape or the scale of the information. In case the location of outliers isn't done properly some time recently information investigation at that point it also may tends to demonstrate mis referencing, one-sided estimation of parameters and wrong results.

It is in this manner related to recognize the outliers prior to continuing advance for analysis and modeling. Much work has been exhausted Univariate probability distributions. The objective of this study is to create strategies which are used to identify the outliers in univariate moment distributions. Two discordancy tests are created to identify the single and two outliers from information characterized by Moment distributions. The precise distribution of created tests does not exist, therefore for finding the Tables of critical values simulation think about was utilized.

The graphs illustrate and distinguish how data behaves across various parameter values. Moment distributions can be biased towards size or area in different research fields. In this study, exponential distribution is employed to detect outliers and assess their impact on distribution parameters. Both size- and area-biased moments are considered. The effectiveness of four outlier detection strategies discussed earlier is evaluated through simulation studies. Tests of various sizes are drawn from different moment distributions, as a single test may not accurately represent a strategy's performance.

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